The 13th Summer School

ANALYSIS, TOPOLOGY and APPLICATIONS

July 29 — August 11, 2018

Vyzhnytsya, Chernivtsi Region, Ukraine

BOOK OF ABSTRACTS

The 13th Summer School "Analysis, Topology and Applications",
July 29 – August 11, 2018
Vyzhnytsya, Chernivtsi Region, Ukraine
Book Of Abstracts

ORGANIZING COMMITTEE

Olena Hryniv (Ivan Franko Lviv National University)
Olena Karlova (Yuriy Fedkovych Chernivtsi National University)
Volodymyr Mykhaylyuk (Yuriy Fedkovych Chernivtsi National University)
Taras Zvozdetskyy (Yuriy Fedkovych Chernivtsi National University)

Contents

On the Mackey topology on abelian topological groups

Lydia Außenhofer

Fakultät für Informatik und Mathematik Universität Passau, Germany

For a locally convex vector space (V, τ) there exists a finest locally convex vector space topology μ such that the topological dual spaces $(V, \tau)'$ and $(V, \mu)'$ coincide algebraically. This topology is called *Mackey topology*. If (V, τ) is a metrizable locally convex vector space, then τ is the Mackey topology.

In 1995 Chasco, Martín Peinador and Tarieladze asked the following question: Given a locally quasi–convex group (G, τ), does there exist a finest locally quasi–convex group topology μ on G such that the character groups $(G, \tau)^\wedge$ and $(G, \mu)^\wedge$ coincide?

Two locally quasi–convex group topologies τ and τ' on an abelian group G are named *compatible* if the character groups $(G, \tau)^\wedge$ and $(G, \tau')^\wedge$ coincide algebraically. The set of group topologies, which are compatible with a given group topology, forms a partially ordered set with the weak topology as bottom element. In case there exists a top element, it is called the *Mackey* topology.

It was shown in the above mentioned paper that every locally compact group topology on an abelian group is the Mackey topology.

In the talk we first examine general result concerning compatible group topologies. Afterwards, we study compatible group topologies on LCA groups. (Surprisingly, the cases of $\mathbb{R}$, $\mathbb{Z}$ and the Prufer groups $\mathbb{Z}(p^\infty)$ are the most difficult ones.)

Then we give a survey on groups where the given topology is / is not the Mackey toplogy. Finally we present a locally quasi–convex group which does not admit a Mackey topology.

References

[1] L. Außenhofer, *On the non–existence of the Mackey topology*, to appear in Forum Math.; https://doi.org/10.1515/forum-2017-0179.

[2] L. Außenhofer, D. de la Barrera Mayoral, D. Dikranjan and E. Martín Peinador, *Varopoulos paradigm: Mackey property vs metrizability in*

topological groups, Revista Matematica Complutense 30 no.1 (2017), 167–184.

[3] L. Außenhofer and D. Dikranjan, *The problem of Mackey topologies: a complete solution in the case of bounded groups*, Topology Appl. 221 (2017) 206–224.

[4] L. Außenhofer, D. Dikranjan and E. Martín Peinador, *Locally quasi-convex compatible topologies on a topological group* , Axioms 4(4) (2015,), 436-458; doi:10.3390/axioms4040436

[5] M.J. Chasco, E. Martín-Peinador and V. Tarieladze, *On Mackey topology for groups*, Stud. Math. 132 no.3 (1999), 257–284.

[6] S. Gabriyelyan, *A locally quasi-convex abelian group without Mackey topology*, preprint 2017, https://arxiv.org/abs/1601.03219v2

e-mail: aussenho@fim.uni-passau.de

On complete topologized semilattices

Taras Banakh

Jan Kochanowski University in Kielce, Poland

and

Ivan Franko National University of Lviv, Ukraine

In the lectures with shall survey some properties of complete topologized semilattices.

A topologized semilattice is a semilattice endowed with a topology.

A topologized semilattice is complete if each chain $C \subset X$ has $\inf C$ and $\sup C$ that belong to the closure of the chain C in X.

We shall prove that each complete topologized semilattice X is closed in any functionally Hausdorff semitopological semilattice that contains X. Also we shall prove that the partial order in a complete functionally Hausdorff semitopological semilattice is a closed subset of $X \times X$.

More details can be found in the following preprints.

References

[1] T. Banakh, S. Bardyla, *Complete topologized posets and semilattices.*

[2] T. Banakh, S. Bardyla, A. Ravsky, *The closedness of complete subsemilattices in functionally Hausdorff semitopological semilattices.*

[3] T. Banakh, S. Bardyla, *On images of complete topologized subsemilattices in sequential semitopological semilattices.*

[4] T. Banakh, S. Bardyla, *The interplay between weak topologies on topological semilattices.*

[5] T. Banakh, S. Bardyla, *Characterizing chain-compact and chain-finite topological semilattices.*

[6] T. Banakh, S. Bardyla, *Completeness and absolute H-closedness of topological semilattices.*

e-mail: tbanakh@gmail.com

Weak topologies on topologized semilattices

Serhii Bardyla

Ivan Franko National University of Lviv, Ukraine

We will discuss an interplay between weak topologies on topologized semilattices.

e-mail: sbardyla@yahoo.com

Curvature properties of LCK-manifolds and their submanifolds

Yevhen Cherevko and **Olena Chepurna**

Odesa National Economic University, Ukraine

Definition 1. *A Hermitian manifold* (M^{2m}, J, g) *is called a* locally conformal Kähler manifold (LCK - manifold) *if there is an open cover* $\mathfrak{U} = \{U_\alpha\}_{\alpha\in A}$ *of* M^{2m} *and a family* $\{\sigma_\alpha\}_{\alpha\in A}$ *of* C^∞ *functions* $\sigma_\alpha : U_\alpha \to \mathbb{R}$ *so that each local metric*

$$\hat{g}_\alpha = e^{-2\sigma_\alpha} g|_{U_\alpha}$$

is Kählerian. An LCK - manifold is endowed with some form ω, so called a Lee form *which can be calculated as [1]*

$$\omega = \frac{1}{m-1}\delta\Omega \circ J,$$

The form should be closed:

$$d\omega = 0.$$

There is A. Gray's[2] classification of Hermitian manifolds : $M^{2m} \in \mathcal{L}_i$ if the curvature tensor R of M^{2m} satisfies the identity (i), where

(1) $R(X,Y,Z,W) = R(X,Y,JZ,JW)$

(2) $R(X,Y,Z,W) =$
$= R(JX,JY,Z,W) + R(JX,Y,JZ,W) + R(JX,Y,Z,JW)$

(3) $R(X,Y,Z,W) = R(JX,JY,JZ,JW)$

Vaisman[3] have proved that if M^{2m} is $LCK-$manifold, then $\mathcal{L}_2 = \mathcal{L}_3$.

It's worth for noting that the classification isn't a complete one. For example, the generalized Hopf manifolds are included in neither class. We study immersion $\Psi : \bar{M}^{2m-1} \longrightarrow M^{2m}$ which is locally represented by functions

$$x^\alpha = x^\alpha(y^1, ..., y^{2m-1}),$$

where $\alpha = 1, ..., 2m,\ \ y^i,\ i = 1, ..., 2m-1$ is a local coordinate system in $\bar{M}^{2p}$. We put here

$$B_i^\alpha = \partial_i x^\alpha.$$

We obtained the theorems.

Theorem 1. *If a hypersurface $\overline{M}^{2m-1}$ of a LCK-manifold M^{2m} is an integral manifold of the distribution defined by the equation*

$$\omega = 0$$

where ω is Lee form of the LCK-manifold M^{2m} that satisfies the condition

$$\nabla_X\omega(Y) = \frac{1}{4}||\omega||^2 g(X,Y) - \frac{1}{2}\omega(X)\omega(Y),$$

i.e. $M^{2m} \in \mathcal{L}_1$, *then induced by the immersion the almost contact structure*

$$\begin{aligned} &1) \quad f_i^j = J_\beta^\alpha B_\alpha^j B_i^\beta; \\ &2) \quad \eta_k = \tfrac{1}{||\omega||} B_k^\beta J_\beta^\alpha \omega_\alpha; \\ &3) \quad \xi^k = -\tfrac{1}{||\omega||} B_\beta^k J_\alpha^\beta \omega^\alpha. \end{aligned}$$

is a c-Sasakian structure, *where* $c = \frac{1}{2}||\omega||$. *Moreover, the manifold* $\overline{M}^{2m-1}$ *is a totally umbilical hypersurface of* M^{2m}.

Theorem 2. *If a hypersurface* $\overline{M}^{2m-1}$ *of an LCK-manifold* M^{2m} *is an integral manifold of the distribution defined by the equation*

$$\omega = 0$$

where ω *is Lee form of the LCK-manifold* M^{2m} *that satisfies the condition*

$$\nabla_X \omega(Y) = \frac{1}{2}||\omega||^2 g(X, Y),$$

then induced by the immersion the almost contact structure

$$\begin{aligned} &1) \quad f_i^j = J_\beta^\alpha B_\alpha^j B_i^\beta; \\ &2) \quad \eta_k = \tfrac{1}{||\omega||} B_k^\beta J_\beta^\alpha \omega_\alpha; \\ &3) \quad \xi^k = -\tfrac{1}{||\omega||} B_\beta^k J_\alpha^\beta \omega^\alpha. \end{aligned}$$

is the cosymplectic structure *i.e. a* normal almost contact metric structure *for which the conditions*

$$1) d\eta = 0, \quad 2) d\bar{\Omega} = 0, \quad 3) n_{ij}^k = 0;$$

are satisfied. Moreover, $\overline{M}^{2m-1}$ *is a totally umbilical hypersurface of* M^{2m}.

Unfortunately, the $LCK-$manifolds whose Lee forms satisfy the equation $\nabla_X \omega(Y) = \frac{1}{2}||\omega||^2 g(X, Y)$ are included in neither class $\mathcal{L}_i$.

References

[1] Kirichenko V. F. *Locally conformally Kählerian manifolds of constant holomorphic sectional curvature*, Mathematics of the USSR-Sbornik (2) **72** (1992), 333-342.

[2] Gray A. *Curvature identities for Hermitian and almost Hermitian manifolds*, Tohoku Math. J. (4) **28** (1976), 601–612.

[3] Vaisman I. *Some Curvature Properties of Locally Conformal Khler Manifolds*, Transactions of the American Mathematical Society (2) **259** (Jun., 1980), 439–447.

e-mail: cherevko@usa.com, chepurna67@gmail.com

Strongly norm attaining Lipschitz operators

Rafael Chiclana Vega

Universidad de Granada, Spain

The study of the density of norm attaining linear operators has its root in the classical Bishop-Phelps theorem which states that the set $\mathrm{NA}(X, \mathbb{R})$ of those linear functionals T from a Banach space X to $\mathbb{R}$ which attain their norm (i.e there exists x in the unit sphere of X such that $|T(x)| = \|T\|$) is dense in the dual space of X. The aim of this talk is to study the set $\mathrm{SNA}(M, Y)$ of those Lipschitz operators from a metric space M to a Banach space Y which attain their Lipschitz norm, as a nonlinear generalisation of the norm attaining linear operators theory. We show some metric spaces M such that $\mathrm{SNA}(M, \mathbb{R})$ is not norm dense and give conditions wich are sufficient to provide the norm density of $\mathrm{SNA}(M, Y)$ in the space of all Lipschitz operators from M to any Banach space Y, studying the relationship between them.

e-mail: rafachiclanavega@gmail.com

About one orthogonal trigonometric Schauder basis for the space $C(\mathbb{T}^2)$

Nadiia Derevianko

Institute of Mathematics of NAS, Ukraine

Let $\mathbb{T}^2 \cong [-\pi, \pi)^2$ be the 2-dimensional torus and $C(\mathbb{T}^2)$ be the space of 2π-periodic in each variable continuous on $\mathbb{R}^2$ functions. Our main result is the construction of an orthogonal trigonometric Schauder basis for the space $C(\mathbb{T}^2)$ [1]. Further for this basis we use notation $\{t_k\}_{k\in\mathbb{N}}$.

Our results generalize the one-dimensional construction that is based on the kernel of de la Vallée Poussin [2]. To construct this basis we use ideas of a dyadic anisotropic periodic multiresolution analysis (PMRA) and corresponding wavelet spaces that were developed in [3] and [4]. The multiresolution analysis is formed using the sequence of only rotation matrices. The polynomial degree is considered in terms of the l_1- and l_∞-norms.

For a function $f \in C(\mathbb{T}^2)$ and $\mu \in \mathbb{N}$ we define the operator

$$S_\mu f = \sum_{k=1}^{\mu} \langle f, t_k \rangle t_k,$$

where $\langle f, t_k \rangle$ are the Fourier coefficients with respect to the basis $\{t_k\}_{k\in\mathbb{N}}$.

The focus of attention is the estimation of the norm $\|S_\mu\|_{C(\mathbb{T}^2)\to C(\mathbb{T}^2)}$.

This is joint work with Vitalii Myroniuk (Institute of Mathematics NAS Ukraine) and Jürgen Prestin (Institut für Mathematik, Universität zu Lübeck).

References

[1] Derevianko N., Myroniuk V., Prestin J. *On an orthogonal bivariate trigonometric Schauder basis for the space of continuous functions*, to appear in J. Approx. Theory.

[2] Prestin J., Selig K. K. *On a constructive representation of an orthogonal trigonometric Schauder basis for $C_{2\pi}$*, Oper. Theory, Adv. Appl., Birkhäuser, Basel. **121** (2001), 402–425.

[3] Langemann D., Prestin J. *Multivariate periodic wavelet analysis*, Appl. Comput. Harm. Anal. **28** (2010), 46-66.

[4] Bergmann R., Prestin J. *Multivariate periodic wavelets of de la Vallée Poussin type*, J. Fourier Anal. Appl. **21** (2015), 342-369.

e-mail: nadyaderevyanko@gmail.com

Categorical resolutions of singular curves

Yuriy Drozd

Institute of Mathematics of National Academy of Sciences, Kyiv, Ukraine

It is a survey of a joint work with Igor Burban.

We construct a categorical resolution of a singular curve using a certain sheaf of orders of finite global dimension. Actually, we show that the derived category of coherent sheaves over this sheaf of orders provides a recollement of the derived category of coherent sheaves over the original curve. In rational case we also show that the resulting derived category is equivalent to the derived category of modules over a quasi-hereditary finite dimesional algebra.

I am also going to explain why the "usual" resolution is not enough, at least from the homological point of view.

e-mail: y.a.drozd@gmail.com

The amenability to algebraic and analytical perspective

Rachid El Harti

University Hassan First Morocco

The talk is a synthesis on my contributions that I have done in several institutes on the *subject of the amenability* to algebraic and analytical perspective since 2003 and is based on the following papers.

References

[1] Contractible Frèchet algebras. Proceedings of the American Mathematical Society. Vol 132, Number 5 Pages: 1251-1255 (2003).

[2] The structure of a subclass of amenable Banach algebras, Int. J. Math. Math. Sc, Volume 2004, 55 (2004) 2963-2969.

[3] Reduction operator algebras and Generalized Similarity Problem, Operators and Matrices, Volume 4, Number 4, 559- 572 (2010).

[4] The semisimplicity of amenable operator algebras. Archiv der Mathematik August 2013, Volume 101, Issue 2, pp 129-133 (with Paulo Pinto).

[5] Amenable Cross product Banach algebras associated with a class of dynamical systems. Integral Equation and Operator Theory. 2017 (with Paulo Pintoand Marcel de Jeu).

e-mail: rachid.elharti@uhp.ac.ma

Can One Hear the Shape of a Group?

Rostislav Grigorchuk
Texas A&M University, USA

e-mail: grigorch@math.tamu.edu

The monoid of monotone injective partial selfmaps of the poset $(\mathbb{N}^3, \leqslant)$ with cofinite domains and images

Oleg Gutik and Olha Krokhmalna
National University of Lviv, Ukraine

We shall follow the terminology of [1].

Let $\mathbb{N}$ be the set of positive integers with the usual linear order $\leq$ and n be an arbitrary positive integer greater then or equal 2. On the Cartesian power $\mathbb{N}^n = \underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{n\text{-times}}$ we define the product partial order, i.e.,

$$(i_1, \ldots, i_n) \leqslant (j_1, \ldots, j_n) \quad \Longleftrightarrow \quad (i_k \leq j_k) \quad \forall\, k = 1, \ldots, n.$$

The set $\mathbb{N}^n$ with this partial order will be denoted by $\mathbb{N}^n_{\leqslant}$.

For an arbitrary positive integer $n \geqslant 2$ by $\mathscr{PO}_{\infty}(\mathbb{N}^n_{\leqslant})$ we denote the *semigroup of injective partial monotone selfmaps of* $\mathbb{N}^n_{\leqslant}$ *with cofinite domains and images.*

We discuss on the semigroup of injective partial monotone selfmaps of $\mathbb{N}^n_{\leqslant}$ with cofinite domains and images. We show that the group of units of the semigroup $\mathscr{PO}_{\infty}(\mathbb{N}^n_{\leqslant})$ is isomorphic to the group of permutations of an n-elements set $\mathscr{S}_n$ and describe the subgroup of idempotents of $\mathscr{PO}_\infty(\mathbb{N}^n_\leqslant)$. In the case $n = 3$ we describe the property of elements of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^n_\leqslant)$ as partial bijections of the poset $\mathbb{N}^3_\leqslant$ and Green's relations on the semigroup $\mathscr{PO}_\infty(\mathbb{N}^3_\leqslant)$. Also the Witt property of the semigroup $\mathscr{PO}_\infty(\mathbb{N}^3_\leqslant)$ is discussed.

References

[1] J. M. Howie, *Fundamentals of Semigroup Theory*, London Math. Monographs, New Ser. 12, Clarendon Press, Oxford, 1995.

e-mails: oleg.gutik@lnu.edu.ua, ovgutik@yahoo.com, Olia709@i.ua

On old and new classes of feebly compact spaces

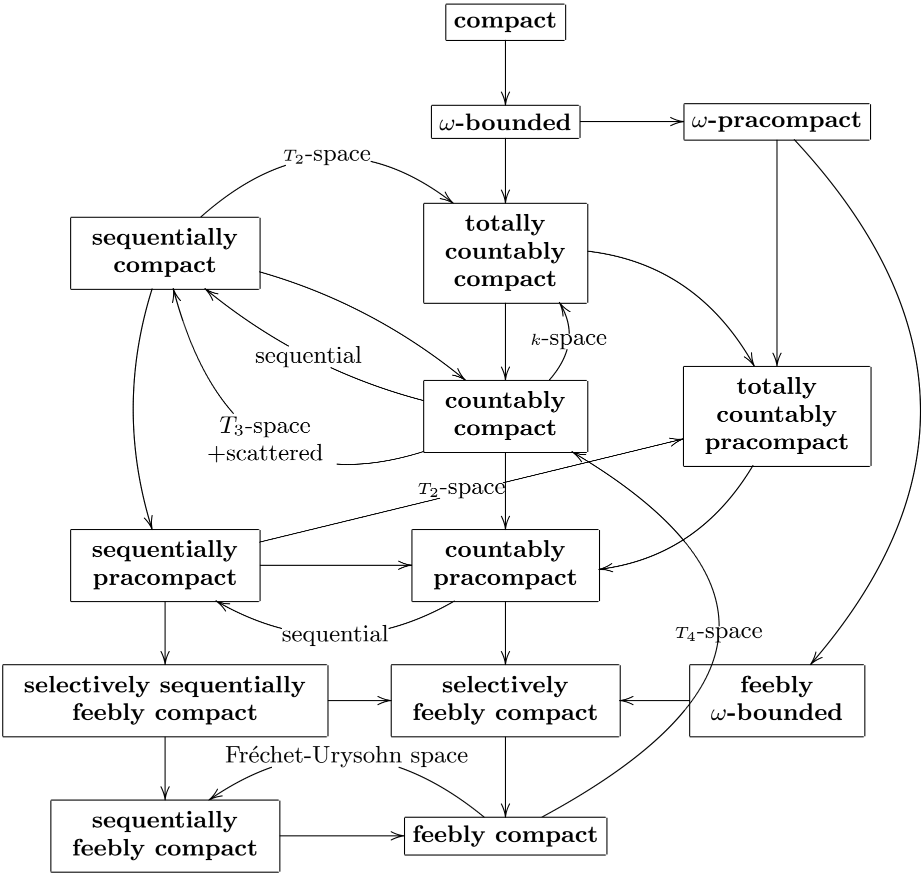
Oleg Gutik and **Alex Ravsky**

Faculty of Mathematics, National University of Lviv, Ukraine and *Pidstryhach Institute for Applied Problems of Mechanics and Mathematics of NASU, Ukraine*

In general topology are often investigated different classes of compact-like spaces and relations between them, see, for instance, basic [3, Chap. 3] and general works [2], [5], [7], [6], [4]. We consider the present paper as a next small step in this quest.

Namely, in order to refine the stratification of compact-like spaces, we introduce three new classes of pracompact spaces, consider their basic properties and relations with other compact-like spaces. Our main motivation to introduce these spaces is possible applications in topological algebra.

Relations between different classes of compact-like spaces are also well-studied. Some of them are presented at Diagram 3 in [5, p.17], at Diagram 1 in [1, p. 58] (for Tychonoff spaces), and at Diagram 3.6 in [6, p. 611]. We also present a big diagram.



References

[1] Dorantes-Aldama A., Shakhmatov D. *Selective sequential pseudocompactness*, Top. Appl. **222** (2017), 53–69.

[2] van Douwen E., Reed G., Roscoe A., Tree I. *Star covering properties* Top. Appl, **39**:1 (1991), 71–103.

[3] Engelking R., *General Topology*, 2nd ed., Heldermann, Berlin, 1989.

[4] Lipparini P., *The equivalence of two definitions of sequential pseudocompactness*, Appl. Gen. Topol. **17**:1 (2016), 1–5. (long version http://arxiv.org/abs/1201.4832)

[5] Matveev M. *A survey on star covering properties*, preprint; (available at http://at.yorku.ca/v/a/a/a/19.htm).

[6] Stephenson Jr. R. *Initially κ-compact and related compact spaces*, in K. Kunen, J. E. Vaughan (eds.), Handbook of Set-Theoretic Topology, Elsevier, 1984, 603–632.

[7] Vaughan J. *Countably compact and sequentially compact spaces*, in K. Kunen, J. E. Vaughan (eds.), Handbook of Set-Theoretic Topology, Elsevier, 1984, 569–602.

e-mails: ovgutik@yahoo.com, alexander.ravsky@uni-wuerzburg.de

Fractal dimensions for non-additive measures

Inna Hlushak and **Oleh Nykyforchyn**

Vasyl' Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine

We follow the terminology and notation of [1] and denote by $\exp X$ the set of all non-empty closed subsets of a compactum X. We call a function $c : \exp X \cup \{\emptyset\} \to I$ a *capacity* on a compactum X if the three following properties hold for all subsets $F, G \underset{\mathrm{cl}}{\subset} X$:

1. $c(\emptyset) = 0$;
2. if $F \subset G$, then $c(F) \le c(G)$ (monotonicity);
3. if $c(F) < a$, then there is an open subset $U \supset F$ such that for all $G \subset U$ the inequality $c(G) < a$ is valid (upper semicontinuity).

If, additionally, $c(X) = 1$ (or $c(X) \le 1$) holds, then the capacity is called *normalized* (resp. *subnormalized*). We denote by MX and $\underline{M}X$ the sets of all normalized and of all subnormalized capacities respectively.

We consider the metric $\hat{d}$ on the set $\underline{M}X$ of subnormalized capacities on a metric compactum (X,d) :

$$\hat{d}(c,c') =$$
$$= \inf\{\varepsilon > 0 \mid c(\bar{O}_\varepsilon(F)) + \varepsilon \geq c'(F), c'(\bar{O}_\varepsilon(F)) + \varepsilon \geq c(F), \forall F \underset{\mathrm{cl}}{\subset} X\},$$

here $\bar{O}_\varepsilon(F)$ is the closed ε-neighborhood of a subset $F \subset X$. This metric determines a compact topology on $\underline{M}X$ [1].

We investigate the problem approximation of arbitrary capacity with regular additive measures on a finite subspace. The article [2] proposes an algorithm which enables to find an additive measure m on subspace $X_0 = \{x_1, x_2, \ldots, x_n\} \subset X$ that is (almost) the closest to capacity $c \in \underline{M}X$ with respect to the distance $\hat{d}$. The presented algorithm is convenient for programmatic implementation but it requires previously calculated values of a capacity for all $2^{\text{cardinality of the space}}$ subsets, which is not appropriate even for ≥ 40 points. Hence, to handle subspaces of greater cardinality, we need investigate dimensional characteristics of capacities.

For a capacity $c \in \underline{M}X$, among all such $X_0 \underset{\mathrm{cl}}{\subset} X$ that $c(A) = c(A \cap X_0)$ for all $A \underset{\mathrm{cl}}{\subset} X$, there is a smallest set, which is called the *support* of c and denoted by $\operatorname{supp} c$.

For a capacity $c \in \underline{M}X$ and a number $\varepsilon \geq 0$, a collection $\mathcal{F}$ of closed sets in X is called an *ε-foundation* of c if, for each non-empty $A \underset{\mathrm{cl}}{\subset} X$, there are $F_1, F_2, \ldots, F_n \in \mathcal{F}$, $n \in \mathbb{N}$, such that $F_i \cap A \neq \varnothing$, for all $1 \leq i \leq n$, and $c(F_1 \cup F_2 \cup \cdots \cup F_n) \geq c(A) - \varepsilon$.

An ε-foundation $\mathcal{F}$ of c is called a *δ-ε-foundation*, for $\varepsilon \geq 0$, $\delta > 0$, if $\operatorname{diam} F \leq \delta$ for all $F \in \mathcal{F}$.

For $c \in \underline{M}X$, $s \geq 0$, $\varepsilon \geq 0$, $\delta > 0$, we denote:

$$N_{\delta,\varepsilon}(c) = \min\bigl\{|\mathcal{F}| \bigm| \mathcal{F} \text{ is a } \delta\text{-}\varepsilon\text{-foundation of } c\bigr\},$$

$$\mathcal{H}^s_{\delta,\varepsilon}(c) = \inf\bigl\{\sum\{(\operatorname{diam} F)^s \mid F \in \mathcal{F}\} \bigm| \mathcal{F} \text{ is a } \delta\text{-}\varepsilon\text{-foundation of } c\bigr\}.$$

We define *the weak upper box dimension*, *the weak lower box di-*

mension, and *the weak Hausdorff dimension* of capacity c:

$$\overline{\dim}_{WB} c = \lim_{\varepsilon \to 0} \overline{\dim}_{\varepsilon B} c,$$
$$\underline{\dim}_{WB} c = \lim_{\varepsilon \to 0} \underline{\dim}_{\varepsilon B} c,$$
$$\dim_{WH} c = \lim_{\varepsilon \to 0} \dim_{\varepsilon H} c,$$

where

$$\overline{\dim}_{\varepsilon B} c = \varlimsup_{\delta \to 0} \frac{\ln N_{\delta,\varepsilon}(c)}{-\ln \delta}, \quad \underline{\dim}_{\varepsilon B} c = \varliminf_{\delta \to 0} \frac{\ln N_{\delta,\varepsilon}(c)}{-\ln \delta},$$

$$\dim_{\varepsilon H} c = \sup\{s \geq 0 \mid \mathcal{H}^s_\varepsilon(c) = \infty\} = \inf\{s \geq 0 \mid \mathcal{H}^s_\varepsilon(c) = 0\}.$$

Proposition 1. *There is a normalized capacity c on I^2 such that $\underline{\dim}_{WB} c = \overline{\dim}_{WB} c < \underline{\dim}_B \operatorname{supp} c = \overline{\dim}_B \operatorname{supp} c$.*

In general, all these dimensions for a capacity $c \in \underline{M}X$ are different (some may be equal) and do not exceed the corresponding classical dimensions [3] of support $\operatorname{supp} c$:

Proposition 2. $\dim_{WH} c \leq \underline{\dim}_{WB} c \leq \overline{\dim}_{WB} c$, $\dim_{WH} c \leq \dim_H \operatorname{supp} c$, $\underline{\dim}_{WB} c \leq \underline{\dim}_B \operatorname{supp} c$, $\overline{\dim}_{WB} c \leq \overline{\dim}_B \operatorname{supp} c$ *for all* $c \in \underline{M}X$.

References

[1] Zarichnyi M.M., Nykyforchyn O.R. *Capacity functor in the category of compacta*, Sb.: Mathematics 2008, **199** (2), 159–184. doi: 10.1070/SM2008v199n02ABEH003914

[2] Nykyforchyn O.R., Hlushak I.D. *Approximation of capacities with additive measures*, Carpathian Mathematical Publications. - 2017. - Vol 9, 1. - P.92-97.

[3] Pesin Ya.B. Dimension Theory in Dynamical Systems: Contemporary Views and Applications, The Chicago University Press, Chicago and London, 1997.

e-mails: inna.hlushak@pu.if.ua, oleh.nyk@gmail.com

Measurable selectors of multifunctions in non-separable spaces

Vladimir Kadets

V.N. Karazin Kharkiv National University, Ukraine

Let X be a real Banach space, and (Ω, Σ, μ) be a complete finite measure space. A *multifunction* is a map $F : \Omega \to 2^X \setminus \{\emptyset\}$, a *selector* of F is a function $f : \Omega \to X$ with $f(t) \in F(t)$ for all $t \in \Omega$.

Theorem 1 (Kuratowski and Ryll-Nardzewski, 1965)**.** *Let X be a separable Banach space, and let $F : \Omega \to 2^X \setminus \{\emptyset\}$ be a multifunction that takes closed values and satisfies the following Effros measurability condition: for each open set $U \subset X$ the subset $\{t \in \Omega : F(t) \cap C \neq \emptyset\}$ belongs to Σ. Then F admits a Borel measurable selector.*

The Kuratowski and Ryll-Nardzewski theorem enables to define an integral of multifunctions through integrals of its selectors, and leads to a number of types of integrals (like strong, weak or Pettis ones) for multifunctions. There are definitions of multifunction integral, that are not based on selectors, but in order to get good properties of that types of integral one also needs some selection theorems. By this reason the multifunctions integration theory deals mostly with multifunctions whose values are subsets of a separable Banach space. Some years ago we started with Bernardo Cascales and José Rodríguez a joint research project whose goal was to build in non-separable spaces an acceptably good theory of integration of multifunctions.

A multi-function $F : \Omega \to 2^X$ is said to be *scalarly measurable*, if for every $x^* \in X^*$ the function $t \mapsto \sup x^*(F(t))$ is measurable. In particular a single valued function $f : \Omega \to X$ is scalarly measurable if the composition $x^* \circ f$ is measurable for every $x^* \in X^*$. I am going to speak about our results about weak types of integrability for multifunctions and scalarly measurable selectors. In particular, the following theorem will be demonstrated.

Theorem 2 (Cascales, Kadets, Rodríguez, 2010)**.** *If all the values of a scalarly measurable multifunction F are weakly compact, then F admits a scalarly measurable selector.*

I strongly believe that the above cited theorem because of its generality may find applications in those areas of applied mathematics where the Kuratowski and Ryll-Nardzewski theorem is often used, and only my ignorance in applied mathematics does not permit me to find such applications.

References

[1] B. Cascales, V. Kadets, J. Rodríguez. *Measurable selectors and set-valued Pettis integral in non-separable Banach spaces*. Journal of Functional Analysis **256**, (2009), No. 3, 673–699.

[2] B. Cascales, V. Kadets, J. Rodríguez. *Measurability and Selections of Multi-Functions in Banach Spaces*. Journal of Convex Analysis **17** (2010), No. 1, 229–240.

[3] B. Cascales, V. Kadets, J. Rodríguez. *The Gelfand Integral for multi-valued functions*. J. Convex Anal. **18** (2011), No. 3, 873-895.

[4] B. Cascales, V. Kadets, J. Rodríguez. *Radon-Nikodyým Theorems for Multimeasures in Non-Separable Spaces*. Journal of Mathematical Physics, Analysis, Geometry 9 (2013), No. 1, 7–24.

[5] D.H.Fremlin *The Cascales-Kadets-Rodríguez selection theorem*.

e-mail: v.kateds@karazin.ua

Fragmentability and related properties

Olena Karlova

Yurii Fed'kovych Chernivtsi National University, Ukraine

Let X be a topological space and (Y, d) be a metric space. A map $f : X \to Y$ is called *ε-fragmented* for some $\varepsilon > 0$ if for every closed nonempty set $F \subseteq X$ there exists a nonempty relatively open set $U \subseteq F$ such that $\mathrm{diam} f(U) < \varepsilon$. If f is ε-fragmented for every $\varepsilon > 0$, then it is called *fragmented*.

Let $\mathscr{U} = (U_\xi : \xi \in [0, \alpha])$ be a transfinite sequence of subsets of a topological space X. We define $\mathscr{U}$ to be *regular in X*, if

(a) each U_ξ is open in X;

(b) $\emptyset = U_0 \subset U_1 \subset U_2 \subset \cdots \subset U_\alpha = X$;

(c) $U_\gamma = \bigcup_{\xi<\gamma} U_\xi$ for every limit ordinal $\gamma \in [0, \alpha)$.

A map $f : X \to Y$ is ε-fragmented iff there exists a regular sequence $\mathscr{U} = (U_\xi : \xi \in [0, \alpha])$ (which is called *ε-associated with f* and is denoted by $\mathscr{U}_\varepsilon(f)$) in X such that $\mathrm{diam} f(U_{\xi+1} \setminus U_\xi) < \varepsilon$ for all $\xi \in [0, \alpha)$.

We say that an ε-fragmented map $f : X \to Y$ is *functionally ε-fragmented* if $\mathscr{U}_\varepsilon(f)$ can be chosen such that every set U_ξ is functionally open in X. Further, f is *functionally ε-countably fragmented* if $\mathscr{U}_\varepsilon(f)$ can be chosen to be countable and f is *functionally countably fragmented* if f is functionally ε-countably fragmented for all $\varepsilon > 0$.

In the lectures we will discuss the relations between functionally countably fragmented maps and other classes of maps (Baire-one, F_σ-measurable, etc.). We will also examine the extension problem of fragmented maps.

e-mail: maslenizza.ua@gmail.com

Bifurcation of cycles in parabolic systems of differential equations with weak diffusion

Ivan Klevchuk

Yurii Fed'kovych Chernivtsi National University, Ukraine

Consider the equation [1, 2]

$$\frac{\partial u}{\partial t} = i\omega_0 u + \varepsilon \left[(\gamma + i\delta) \frac{\partial^2 u}{\partial x^2} + (\alpha + i\beta) u \right] + (d_0 + ic_0) u^2 \overline{u} \qquad (1)$$

with periodic boundary condition

$$u(t, x + 2\pi) = u(t, x), \qquad (2)$$

where ε is a small positive parameter.

Theorem 1. *If $\omega_0 > 0$, $\alpha > 0$, $\gamma > 0$, $d_0 < 0$ and the condition $\alpha > \gamma n^2$ is satisfied for some $n \in \mathbb{Z}$. Then for some $\varepsilon_0 > 0$, $0 < \varepsilon < \varepsilon_0$, the periodic on t solutions*

$$u_n = u_n(t, x) = \sqrt{\varepsilon} r_n \exp(i(\chi_n(\varepsilon) t + nx)) + O(\varepsilon)$$

of the boundary value problem (1), (2) exists. Here

$$r_n = \sqrt{(\alpha - n^2\gamma)\,|d_0|^{-1}},$$

$$\chi_n(\varepsilon) = \omega_0 + \varepsilon\beta + \varepsilon c_0 r_n^2 - \varepsilon\delta n^2,$$

$n \in \mathbb{Z}$.

These solutions are exponentially orbitally stable if and only if the condition

$$(d_0 r_n^2 - \gamma k^2)^2(\gamma^2 k^2 + \delta^2 k^2 - 2\gamma d_0 r_n^2 - 4\gamma^2 n^2 - 2\delta c_0 r_n^2) > 4\gamma^2 n^2 (c_0 r_n^2 - \delta k^2)^2$$

is satisfied for all $k \in \mathbb{Z}\backslash\{0\}$.

Consider the equation of spin combustion

$$\frac{\partial^2\xi}{\partial t^2} + \xi = 2\varepsilon\left[\frac{\partial\xi}{\partial t}\left(1 - \frac{4}{3}\left(\frac{\partial\xi}{\partial t}\right)^2\right) + \frac{1}{\varrho^2}\frac{\partial^3\xi}{\partial t\partial x^2}\right], \quad \xi(t, x+2\pi) = \xi(t,x), \tag{3}$$

where ε is a small positive parameter, $\varrho > 0$. There exist the traveling waves $\xi_n(t,x) = \sqrt{1 - \dfrac{n^2}{\varrho^2}}\cos(t + nx) + O(\varepsilon)$ of the problem (3), where $n \in \mathbb{Z}$, $n^2 < \varrho^2$. The traveling waves $\xi_n(t,x)$ are exponentially orbitally stable if and only if the conditions $n^2 < \dfrac{1}{6}(2\varrho^2 + 1)$ are fulfilled.

Consider the Brusselator equations with weak diffusion

$$\frac{\partial u_1}{\partial t} = A - (B+1)u_1 + u_1^2 u_2 + \varepsilon d_1 \frac{\partial^2 u_1}{\partial x^2},$$

$$\frac{\partial u_1}{\partial t} = Bu_1 - u_1^2 u_2 + \varepsilon d_2 \frac{\partial^2 u_2}{\partial x^2}$$

with periodic boundary conditions

$$u_1(t, x+2\pi) = u_1(t,x), \quad u_2(t, x+2\pi) = u_2(t,x),$$

where $B = 1 + A^2 + \varepsilon$, $d_1 > 0$, $d_2 > 0$, ε is a small positive parameter. If $A > 0$ and the condition $1 > (d_1 + d_2)n^2$ is satisfied for some $n \in \mathbb{Z}$, then for some $\varepsilon_0 > 0$, $0 < \varepsilon < \varepsilon_0$, the periodic on t solutions

$u_1 + iu_2 = \sqrt{\varepsilon} r_n \exp(i(At + nx)) + O(\varepsilon)$ of the Brusselator model exists. Here $r_n^2 = \dfrac{4A^2}{A^2+2}\left(1 - (d_1 + d_2)n^2\right)$, $n \in \mathbb{Z}$.

Let $\mathbb{R}^n$ be n-dimensional space with the norm $|u| = \sqrt{u_1^2 + ... + u_n^2}$, $\mathbb{C} = \mathbb{C}[-\Delta, 0]$ be the space of functions, continuous on $[-\Delta, 0]$ with values in $\mathbb{R}^n$ with the norm $||\varphi|| = \sup\limits_{-\Delta \le \vartheta \le 0} |\varphi(\vartheta)|$. We denote by u_t the element of space $\mathbb{C}$ defined by function $u_t(\vartheta, x) = u(t + \vartheta, x)$, $-\Delta \le \vartheta \le 0$.

We consider a parabolic system with delay and weak diffusion

$$\frac{\partial u}{\partial t} = \varepsilon D \frac{\partial^2 u}{\partial x^2} + L(\varepsilon) u_t + f(u_t, \varepsilon) \tag{4}$$

with periodic condition (2). Here ε is a small positive parameter, $u \in \mathbb{R}^n$, $L(\varepsilon) : \mathbb{C} \to \mathbb{R}^n$ is a continuous linear operator, $f : \mathbb{C} \times [0, \varepsilon_0) \to \mathbb{R}^n$, $f(\varphi, \varepsilon) = O(\|\varphi\|^2)$ as $\|\varphi\| \to 0$, the operator $f(\varphi, \varepsilon)$ is continuous in ε and four times continuously differentiable in φ. Let us assume that the zero solution of (4) for $\varepsilon = 0$ is asymptotically stable. The existence and stability of an arbitrarily large finite number of cycles for the equation (4) were considered in [2].

References

[1] Klevchuk I.I. *Existence of countably many cycles in hyperbolic systems of differential equations with transformed argument*, J. Math. Sci. **215**:3 (2016), 341–349.

[2] Klevchuk I.I. *Bifurcation of self-excited vibrations for parabolic systems with retarded argument and weak diffusion*, J. Math. Sci. **226**:3 (2017), 285–295.

e-mail: i.klevchuk@chnu.edu.ua

R-spaces and uniform limits of sequences of functions

Mykhaylo Lukan

Yurii Fed'kovych Chernivtsi National University, Ukraine

A metric space (Y, d) is said to be an *R-space*, if for every $\varepsilon > 0$ there exists a continuous map $r_\varepsilon : Y \times Y \to Y$ with the following properties:

$$d(y,z) \le \varepsilon \implies r_\varepsilon(y,z) = y,$$
$$d(r_\varepsilon(y,z),z) \le \varepsilon$$

for all $y, z \in Y$.

Every convex subset Y of a normed space $(Z, \|\cdot\|)$ equipped with the metric induced from $(Z, \|\cdot\|)$ is an R-space, where the map r_ε is defined as

$$r_\varepsilon(y,z) = \begin{cases} z + (\varepsilon/\|y-z\|)\cdot(y-z), & \|y-z\| > \varepsilon, \\ y & \text{otherwise.} \end{cases}$$

Let X and Y be topological spaces. A map $f : X \to Y$ is *a Baire-one map*, if there is a sequence $(f_n)_{n=1}^\infty$ of continuous maps between X and Y such that $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in X$. The collection of all Baire-one maps between X and Y is denoted by $\mathrm{B}_1(X,Y)$.

It is well-known [1] that the uniform limit of a sequence of Baire-one maps $f_n : X \to \mathbb{R}$ belongs to the first Baire class. Since the uniform limit of a sequence of continuous maps $f_n : X \to Y$ is a continuous map for any metric space Y, it is natural to ask whether the similar fact is true for Baire-one maps. It was proved in [2] that there exist a subset of $Y \subseteq \mathbb{R}^2$ and a sequence of Baire-one functions $f_n : [0,1] \to Y$ which converges uniformly to a function $f \notin \mathrm{B}_1([0,1],Y)$. However, R-spaces are favorable range spaces for the problem on uniform limit of Baire-one functions.

Theorem 1. *Let X be a topological space, (Y,d) be a metric R-space. Then the class $\mathrm{B}_1(X,Y)$ is closed under uniform limits.*

A result in the same direction was proved in [2].

Theorem 2. *Let X be a topological space, (Y,d) be a metric path-connected and locally path-connected space. Then the class $\mathrm{B}_1(X,Y)$ is closed under uniform limits.*

Therefore, it is actual to find relations between R-spaces and path-connected locally path-connected spaces.

Theorem 3. *Every path-connected R-space is locally path-connected.*

Theorem 4. *The unit circle $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ is an R-space.*

Question 1. *Is Hawaiian Earring an R-space?*

References

[1] Kuratowski K. *Topology, Volume 1*, Academic Press (1966).

[2] Karlova O., Mykhaylyuk V. *Functions of the first Baire class with values in metrizable spaces*, Ukr. Math. J., Volume 58, Issue 4, (2006), 567–571.

Cauchy completeness for metric spaces and enriched categories

Volodymyr Lyubashenko

Institute of Mathematics of National Academy of Sciences, Kyiv, Ukraine

We introduce the notion of a category enriched in a monoidal category [1] via the example of metric spaces. Generalized metric spaces of Lawvere [2] are discussed in detail. For them Cauchy completeness (in the sense of Lawvere [2]) is equivalent to the property that every fundamental sequence converges to at least one point. For ordinary categories (enriched in Set) Cauchy completeness is equivalent to the property that every idempotent splits. Denote by Ab the monoidal category of abelian groups. An Ab-category is Cauchy complete iff it admits finite direct sums and every idempotent splits. Thus, the notion of completeness relates analysis and algebra.

References

[1] Gregory Maxwell Kelly, *Basic concepts of enriched category theory*, Theory Appl. Categ. **64** (1982, 2005), no. 10, vi+137 pp. http://www.tac.mta.ca/tac/reprints/articles/10/tr10abs.html Reprint of the 1982 original [Cambridge Univ. Press, Cambridge].

[2] F. William Lawvere, *Metric spaces, generalised logic, and closed categories*, Rendiconti del Seminario Matematico e Fisico di Milano, vol. 43, Tipografia Fusi, Pavia, 1973.

e-mail: lub@imath.kiev.ua

Homotopy properties of spaces of smooth functions on surfaces

Sergiy Maksymenko

Institute of Mathematics of National Academy of Sciences, Ukraine

Let M be a smooth (C^∞) connected compact surface without boundary. Consider the action of the group of C^∞ diffeomorphisms $D(M)$ on the space $C^\infty(M)$ of smooth functions on M by the rule: the result of the right action of $h \in D(M)$ on $f \in C^\infty(M)$ is the composition $f \circ h$. Then for each $f \in C^\infty(M)$ one can define its stabilizer

$$S(f) = \{h \in D(M) \mid f \circ h = f\},$$

and the orbit

$$O(f) = \{f \circ h \mid h \in D(M)\}.$$

Endow $D(M)$ and $C^\infty(M)$ with C^∞ Whitney topologies. Then these topologies yield certain topologies on the corresponding stabilizes and orbits of smooth functions $f \in C^\infty(M)$. Denote by $S_{id}(f)$ the identity path component of $S(f)$, and by $O_f(f)$ the path component of $O(f)$ containing f.

The aim these lectures is to describe the homotopy types of the connected components of $S(f)$ and $O(f)$ for the case when f is a Morse function.

Let $f : M \to \mathbb{R}$ be a C^∞ Morse function. Then the following satements hold.

1. Suppose f has at least one saddle critical point. Then

   - $S_{id}(f)$ is contractible;
   - there exists a free action of a finite group G on a k-torus $T^k = \underbrace{S^1 \times \cdots \times S^1}$ such that $O_f(f)$ is homotopy equivalent to the quotient T^k/G.

2. Otherwise, when f has no saddles,

   - $S_{id}(f)$ is homotopy equivalent to the circle;

- there exists a free action of a finite group G on some T^k such that $O_f(f)$ is homotopy equivalent to $SO(3) \times T^k/G$.

We will also describe the geometrical meaning of the group G, the number k and precise algebraic structure of the fundamental group $\pi_1 O_f(f)$ for the case when M is orientable and disinct from S^2.

In fact, the obtained results also hold for a more general class maps f from compact surfaces (possibly with boundary) to $\mathbb{R}$ or to the circle S^1.

References

[1] E. Kudryavtseva, *Special framed Morse functions on surfaces*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. (2012), no. 4, 14–20.

[2] E. Kudryavtseva, *The topology of spaces of Morse functions on surfaces*, Math. Notes **92** (2012), no. 1-2, 219–236, Translation of Mat. Zametki **92** (2012), no. 2, 241–261.

[3] E. Kudryavtseva, *On the homotopy type of spaces of Morse functions on surfaces*, Sb. Math. **204** (2013), no. 1, 75–113.

[4] E. Kudryavtseva, *Topology of spaces of functions with prescribed singularities on the surfaces*, Doklady Math. **93** (2016), no. 3, 264–266.

[5] S. Maksymenko, *Homotopy types of stabilizers and orbits of Morse functions on surfaces*, Ann. Global Anal. Geom. **29** (2006), no. 3, 241–285.

[6] S. Maksymenko, *Homotopy types of right stabilizers and orbits of smooth functions on surfaces*, Ukrainian Math. Journal **64** (2012), no. 9, 1186–1203 (Russian), `arXiv:math/1205.4196`

[7] S. Maksymenko and B. Feshchenko, *Homotopy properties of spaces of smooth functions on 2-torus*, Ukrainian Math. Journal **66** (2014), no. 9, 1205–1212 (Russian).

[8] S. Maksymenko and B. Feshchenko, *Orbits of smooth functions on* 2-*torus and their homotopy types*, Matematychni Studii **44** (2015), no. 1, 67–84.

[9] S. Maksymenko and B. Feshchenko, *Smooth functions on 2-torus whose Kronrod-Reeb graph contains a cycle*, Methods Funct. Anal. Topology **21** (2015), no. 1, 22–40.

[10] S. Maksymenko, *Deformations of functions on surfaces by isotopic to the identity diffeomorphisms* `arXiv:1311.3347`

e-mail: maks@imath.kiev.ua

On variants of the extended bicyclic semigroup

Kateryna Maksymyk and **Oleg Gutik**
Ivan Franko National University of Lviv, Ukraine

We shall follow the terminology of [3, 10]. In our report all spaces are assumed to be Hausdorff.

A *(semi)topological semigroup* is a topological space with a (separately) continuous semigroup operation. A topology τ on a semigroup S is called:

- *shift-continuous* if (S, τ) is a semitopological semigroup;
- *semigroup* if (S, τ) is a topological semigroup.

The *bicyclic semigroup* (or the *bicyclic monoid*) $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by two elements p and q subject only to the condition $pq = 1$. The bicyclic monoid $\mathscr{C}(p,q)$ is a combinatorial bisimple F-inverse semigroup and it plays an important role in the algebraic theory of semigroups and in the theory of topological semigroups.

The bicyclic semigroup admits only the discrete semigroup topology and if a topological semigroup S contains it as a dense subsemigroup then $\mathscr{C}(p,q)$ is an open subset of S [4]. Bertman and West in [1] extend this result for the case of Hausdorff semitopological semigroups. Amazing dichotomy for the bicyclic monoid with adjoined zero $\mathscr{C}^0 = \mathscr{C}(p,q) \sqcup \{0\}$ was proved in [7]: every Hausdorff locally compact semitopological bicyclic semigroup with adjoined zero $\mathscr{C}^0$ is either compact or discrete.

An interassociate of a semigroup $(S, \cdot)$ is a semigroup $(S, *)$ such that for all $a, b, c \in S$, $a \cdot (b * c) = (a \cdot b) * c$ and $a * (b \cdot c) = (a * b) \cdot c$. This definition of interassociativity was studied extensively in 1996 by Boyd, Gould, and Nelson in [2]. Certain classes of semigroups are known to give rise to interassociates with various properties. For example, it is very easy to show that if S is a monoid, every interassociate must satisfy the condition $a * b = a \cdot c \cdot b$ for some fixed element $c \in S$ (see [2]). This type of interassociate was called a *variant* by

Hickey [9]. A general theory of variants has been developed by a number of authors. In the paper [6] the bicyclic semigroup $\mathscr{C}(p,q)$ and its interassociates $(\mathscr{C}(p,q), *_{m,n})$ are investigated.

By $\mathbb{Z}$ we denote the sets of all integers. On the Cartesian product $\mathscr{C}_{\mathbb{Z}} = \mathbb{Z} \times \mathbb{Z}$ we define the semigroup operation as follows:

$$(a,b) \cdot (c,d) = \begin{cases} (a-b+c, d), & \text{if } b \leqslant c; \\ (a, d+b-c), & \text{if } b > c, \end{cases}$$

for $a,b,c,d \in \mathbb{Z}$. The set $\mathscr{C}_{\mathbb{Z}}$ with such defined operation is called the *extended bicyclic semigroup* [11]. In the paper [5] algebraic properties of $\mathscr{C}_{\mathbb{Z}}$ were described and it was proved therein that every non-trivial congruence $\mathfrak{C}$ on the semigroup $\mathscr{C}_{\mathbb{Z}}$ is a group congruence, and moreover the quotient semigroup $\mathscr{C}_{\mathbb{Z}}/\mathfrak{C}$ is isomorphic to a cyclic group. Also it was shown that the semigroup $\mathscr{C}_{\mathbb{Z}}$ as a Hausdorff semitopological semigroup admits only the discrete topology and the closure $\mathrm{cl}_T(\mathscr{C}_{\mathbb{Z}})$ of the semigroup $\mathscr{C}_{\mathbb{Z}}$ in a topological semigroup T was studied.

In the paper [8] we studied semitopological interassociates

$$(\mathscr{C}(p,q), *_{m,n})$$

of the bicyclic monoid $\mathscr{C}(p,q)$ for arbitrary non-negative integers m and n. In particular, we showed that for arbitrary non-negative integers m, n and every Hausdorff topology τ on $\mathscr{C}_{m,n}$ such that $(\mathscr{C}_{m,n}, \tau)$ is a semitopological semigroup, is discrete. Also, we proved that if an interassociate of the bicyclic monoid $\mathscr{C}_{m,n}$ is a dense subsemigroup of a Hausdorff semitopological semigroup $(S, \cdot)$ and $I = S \setminus \mathscr{C}_{m,n} \neq \varnothing$ then I is a two-sided ideal of the semigroup S and show that for arbitrary non-negative integers m, n, any Hausdorff locally compact semitopological semigroup $\mathscr{C}^0_{m,n}$ ($\mathscr{C}^0_{m,n} = \mathscr{C}_{m,n} \sqcup \{0\}$) is either discrete or compact.

We describe the group $\mathbf{Aut}(\mathscr{C}_{\mathbb{Z}})$ of automorphisms of the extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}$ and study a variant $\mathscr{C}_{\mathbb{Z}}^{m,n} = (\mathscr{C}_{\mathbb{Z}}, *_{m,n})$ of the extended bicycle semigroup $\mathscr{C}_{\mathbb{Z}}$, where $m,n \in \mathbb{Z}$, which is defined by the formula

$$(a,b) *_{m,n} (c,d) = (a,b) \cdot (m,n) \cdot (c,d).$$

We prove that $\mathbf{Aut}(\mathscr{C}_{\mathbb{Z}})$ is isomorphic to the additive group of integers, the extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}$ and every its variant are not finitely generated, describe the subset of idempotents $E(\mathscr{C}_{\mathbb{Z}}^{m,n})$ and Green's relations on the semigroup $\mathscr{C}_{\mathbb{Z}}^{m,n}$, and show that any two variants of the extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}$ are isomorphic. At the end we discuss shift-continuous Hausdorff topologies on the variant $\mathscr{C}_{\mathbb{Z}}^{0,0}$. In particular, we prove that if τ is a Hausdorff shift-continuous topology on $\mathscr{C}_{\mathbb{Z}}^{0,0}$ then each of inequalities $a>0$ or $b>0$ implies that (a,b) is an isolated point of $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau\right)$ and constructe an example a Hausdorff semigroup topology τ^* on the semigroup $\mathscr{C}_{\mathbb{Z}}^{0,0}$ such that all its points with the properties $ab\leqslant 0$ and $a+b\leqslant 0$ are not isolated in $\left(\mathscr{C}_{\mathbb{Z}}^{0,0},\tau^*\right)$.

References

[1] M. O. Bertman, T. T. West, *Conditionally compact bicyclic semitopological semigroups*, Proc. Roy. Irish Acad. **A76** (1976), 219–226.

[2] S. J. Boyd, M. Gould, and A. Nelson, *Interassociativity of semigroups*, Misra, P. R. (ed.) et al., Proceedings of the Tennessee Topology Conference, Nashville, TN, USA, June 10–11, 1996. Singapore, World Scientific, (1997), pp. 33–51.

[3] A. H. Clifford, G. B. Preston, *The algebraic theory of semigroups*, Vols. I and II, Amer. Math. Soc. Surveys **7**, Providence, R.I., 1961 and 1967.

[4] C. Eberhart, J. Selden, *On the closure of the bicyclic semigroup*, Trans. Amer. Math. Soc. **144** (1969), 115–126.

[5] I. R. Fihel, O. V. Gutik, *On the closure of the extended bicyclic semigroup*, Carpathian Math. Publ. **3** (2011), 131–157.

[6] B. N. Givens, A. Rosin, K. Linton, *Interassociates of the bicyclic semigroup*, Semigroup Forum **94** (2017), 104–122.

[7] O. Gutik, *On the dichotomy of a locally compact semitopological bicyclic monoid with adjoined zero*, Visn. Lviv. Univ., Ser. Mekh.-Mat. **80** (2015), 33–41.

[8] O. Gutik, K. Maksymyk, *On semitopological interassociates of the bicyclic monoid*, Visn. Lviv. Univ., Ser. Mekh.-Mat. **82** (2016), 98–108.

[9] J. B. Hickey, *Semigroups under a sandwich operation*, Proc. Edinb. Math. Soc., II. Ser. **26** (1983), 371–382.

[10] W. Ruppert, *Compact semitopological semigroups: an intrinsic theory*, Lect. Notes Math., **1079**, Springer, Berlin, 1984.

[11] R. J. Warne, *I-bisimple semigroups*, Trans. Amer. Math. Soc. **130** (1968), 367–386.

e-mails: kate.maksymyk15@gmail.com, oleg.gutik@lnu.edu.ua

Superfractality of an incomplete sums set of a certain positive series

Vita Markitan and **Ihor Savchenko**

National Pedagogical Dragomanov University, Ukraine

The achievement set (or a *partial sumset*) of the series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + ... + a_n + ... = S_n + (a_{n+1} + a_{n+2} + ...) = S_n + r_n$$

is called a set

$$E\{a_n\} \equiv \left\{ x: \; x = x(M) = \sum_{n \in M \subset \mathbb{N}} a_n, \;\; M \in 2^{\mathbb{N}} \right\},$$

and every its element – *an incomplete sum* of the series.

We consider a family of convergent positive normed series with real terms defined by conditions

$$\sum_{n=1}^{\infty} d_n = \underbrace{c_1 + ... + c_1}_{a_1} + \underbrace{c_2 + ... + c_2}_{a_2} + \ldots + \underbrace{c_n + ... + c_n}_{a_n} + \widetilde{r}_n = 1, \quad (1)$$

where (a_n) is a nondecreasing sequence of real numbers. For a partial case $(a_n) = 2^{n-1}$, $c_n = (n+1)\widetilde{r}_n$, $n \in \mathbb{N}$, the geometry of a series (i.e. properties of cylindrical sets, metric relations generated by them, and topological and metric properties of the set of all incomplete sums of a series) are invistigated.

We consider a random variable

$$\xi = \sum_{n=1}^{\infty} d_n \xi_n, \quad (2)$$

where (ξ_n) — sequence of independent random variables with distribution:

$$P\{\xi_n = 0\} = p_{0n} \geq 0, \quad P\{\xi_n = 1\} = p_{1n} \geq 0, \quad p_{0n} + p_{1n} = 1. \quad (3)$$

Theorem 1. *The distribution of the random variable 2, defined by the series 1, is pure. It is purely discrete if and only if*

$$M = \prod_{k=1}^{\infty} max\{p_{0k}, p_{1k}\} > 0. \tag{4}$$

In the case of discreteness of the distribution of the random variable 2 its point spectrum consists of a point

$$x_0 = \sum_{n=1}^{\infty} \alpha_n^* d_n, \qquad p_{\alpha_n^* n} \geq p_{[1-\alpha_n^*]n},$$

and all points x which can be represented in the form:

$$x = \sum_{n=1}^{m} \alpha_n d_n + \sum_{n=m+1}^{\infty} \alpha_n^* d_n,$$

where $\alpha_n \in \{0, 1\}$, $p_{\alpha_n n} \neq 0$ for $n \leq m$.

By the *spectrum* S_ξ of the distribution of a random variable ξ we mean the set of growth points of its distribution function $F_\xi(x)$, i.e.

$$S_\xi = \{x : F_\xi(x+\varepsilon) - F_\xi(x-\varepsilon) = P\{\xi \in (x-\varepsilon; x+\varepsilon)\} > 0, \forall \varepsilon > 0\}.$$

Lemma 2. *If $p_{in} > 0$ for all $i \in \{0, 1\}$ and all $n \in \mathbb{N}$, then the spectrum S_ξ of the distribution of the random variable ξ coincides with the set $E\{d_n\}$ of all subsums (incomplete sums) of the series* (1), *that is,*

$$S_\xi = E\{d_n\} \equiv \left\{ x : x = \sum_{n \in M} d_n, \quad M \in 2^{\mathbb{N}} \right\}.$$

Corollary 3. *For the spectrum S_ξ of the distribution of a random variable ξ the following inclusion holds: $S_\xi \subset E\{d_n\}$.*

Lebesgue measure zero subsets of $\mathbb{R}$ with the Haussdorff-Besikovich dimension 1, are called *superfractal*.

Theorem 4. *The set of incomplete sums* (1) *is a superfractal set.*

Corollary 5. *The spectrum S_ξ of the distribution of a random variable ξ is a superfractal set.*

Theorem 6. *In the case of continuity ($M = 0$), the distribution of the random variable ξ is a singular Cantor type distribution with a superfractal spectrum.*

An autoconvolution of the distribution of a random variable ξ is the distribution of the random variable $\psi_2 = \xi^{(1)} + \xi^{(2)}$, and an *$s$-multiple convolution of the distribution of a random variable ξ* is the distribution of a random variable

$$\psi_s = \xi^{(1)} + \xi^{(2)} + ... + \xi^{(s)},$$

where $\xi^{(j)}$ are independent and equally distributed random variables each having the same distribution as ξ.

Lemma 7. *The spectrum S_{ψ_s} of the distribution of a random variable ψ_s is a subset of the interval $[0, s]$ and belongs to the union $\prod\limits_{k=0}^{n} (s \cdot 2^k + 1)$ of isometric segments of length $s\widetilde{r}_n$, $n = 1, 2, 3, \ \ldots$.*

Theorem 8. *In the case of the continuity of the random variable ξ ($M = 0$) the distribution of the random variable ψ_s, for any natural $s \geq 2$ is a singular Cantor type distribution with a superfractal spectrum.*

References

[1] Kakeya S. *On the partial sums of an infinite series* // Tôhoku Sci Rep. — 1914. —Vol. **3**, N 4. — P. 159–163.

[2] Peres Y., Schlag W., Solomyak B. *Sixty year of Bernoulli convolutions* // Fractal Geometry and Stochastics II. Progress in Probobility.— 2000. —Vol. **46**. — P. 39–65.

[3] Pratsiovytyi M. V., Savchenko I. O. *Distributions of random incomplete sums of a positive series with a nonlinear property of homogeneity* // Theory of Probability and Mathematical Statistics. — 2014. — **91**. — P. 133-142.

e-mail: v.p.markitan@npu.edu.ua

On invariant max-plus closed convex sets of idempotent measures

Natalia Mazurenko[1] and **Mykhailo Zarichnyi**[2]

[1] *Vasyl Stefanyk Precarpatian National University, Ukraine*

[2] *University of Rzeszów, Poland*

In [4], the authors defined the notion of an invariant idempotent measure under given IFS on a complete metric space. These measures are idempotent counterparts of the probabilistic fractals [1].

In [2], [3], the first-named author considered invariant objects under given IFS on a complete ultrametric space, and obtained the construction of an invariant max-plus closed convex set of idempotent measures.

Recall that an idempotent measure on a compact Hausdorff space X is a functional $\mu\colon C(X) \to \mathbb{R}$ that preserves constants, the maximum operation and is weakly additive (i.e., preserves sums in which at least one summand is a constant function) [5]. Given an arbitrary metric space X, we denote by $I(X)$ the set of idempotent measures of compact support on X. A nonempty subset $A \subset I(X)$ is called max-plus convex if $\max\{t + \mu, \nu\} \in A$ for every $\mu, \nu \in A$ and $t \in [-\infty, 0]$.

The aim of the talk is to obtain a counterpart for the construction of an invariant max-plus closed convex set of idempotent measures under given IFS on a complete metric space. For a complete metric space X, we consider the space $ccI(X)$ (of closed convex subsets of idempotent measures of compact support on X) as a subspace of $C'(X)$ and use the weak* convergence to prove the existence of an invariant element.

References

[1] J. E. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J. 30(1981), 713–747.

[2] N. Mazurenko, *On ultrametric fractals generated by closed convex sets of measures*, Proc. of XI Summer School "Algebra, Topology, and Analysis", 1-14 August 2016, Odesa, - P. 98

[3] N. Mazurenko, *On ultrametric fractals generated by max-plus closed convex sets of idempotent measures*, Proc. Intern. Conf. "Modern advances in Geometry and Topology". - Kharkiv. - 2016. - P. 34.

[4] N. Mazurenko, M. Zarichnyi, *Invariant idempotent measures*, Carpathian Math. Publ. 2018, 10 (1), P. 10–16

[5] M. Zarichnyi, *Spaces and maps of idempotent measures*, Izvestiya: Mathematics, 2010, 74:3, 481–499.

e-mails: mnatali@ukr.net, zarichnyi@yahoo.com

Development of Hahn's theorem on intermediate function

Vasyl Melnyk and **Volodymyr Maslyuchenko**

Yuriy Fedkovych Chernivtsi National University, Ukraine

We call a pair of functions (g, h) *a Hahn's pair* on topological space X, if $g : X \to \mathbb{R}$ and $h : X \to \mathbb{R}$ are upper and lower semicontinuous functions such that $g(x) \leq h(x)$ on X. If $g(x) < h(x)$ on X, then (g, h) is *a strict Hahn's pair.* Function $f : X \to \mathbb{R}$ is called *intermediate* for Hahn's pair (g, h) on X, if $g(x) \leq f(x) \leq h(x)$ on X and *strictly intermediate*, if $g(x) < f(x) < h(x)$ whenever $g(x) < h(x)$ and $g(x) = f(x) = h(x)$ when $g(x) = h(x)$. According to Hahn-Dieudonne-Katetov-Tong theorem [1, p. 105] T_1-space X is normal iff each Hahn's pair (g, h) on X has an intermediate continuous function. This theorem has a lot of analogues (see [2] and the literature given there). Here we present some of our recent results regarding differentiable intermediate function and intermediate function for Hahn's pair of separately semicontinuous functions.

Theorem 9. *Let X be a separable Hilbert space and (g, h) be a Hahn's strict pair on X. Then there is a C^∞-function $f : X \to \mathbb{R}$ which is strictly intermediate for (g, h).*

This result is also true for Asplund spaces and parallelepipeds in $\mathbb{R}^n$

For a map $f : X \times Y \to Z$ and a point $(x, y) \in X \times Y$ we write $f^x(y) = f(x, y) = f_y(x)$. For topological spaces X, Y and Z we denote by $C(X)$, $C^u(X)$ and $C^l(X)$ the spaces of continuous and upper or lower respectively semicontinuous functions $f : X \to \mathbb{R}$, by $CC(X \times Y)$, $C^uC^u(X \times Y)$ and $C^lC^l(X \times Y)$ — spaces of separately continuous and upper or lower separately semicontinuous functions $f : X \times Y \to \mathbb{R}$, and by $C(X, Y)$ and $CC(X \times Y, Z)$ — spaces of continuous maps $f : X \to Y$ and separately continuous maps $f : X \times Y \to Z$, respectively.

For topological spaces X and Y the ordered pair (g, h) of functions $g \in C^uC^u(X \times Y)$ and $h \in C^lC^l(X \times Y)$ is called a *separate Hahn's pair*.

Let us recall, that *plus-topology* $\mathcal{C}$ on product $X \times Y$ of two topological spaces consists of sets $O \subseteq X \times Y$ such that for each point $p = (x, y) \in O$ there exist neighborhoods U of point x and V of point y in spaces X and Y, respectively, with $(U \times \{y\}) \cup (\{x\} \times V) \subseteq O$.

Theorem 10. *Let X and Y be T_1-spaces. Than each separate Hahn's pair (g, h) on product $X \times Y$ has an intermediate separately continuous function if and only if space $Q = (X \times Y, \mathcal{C})$ is normal.*

References

[1] R. Engelking, *General Topology*, Revised and completed edition. Heldermann Verlag, Berlin, (1989).

[2] V. Maslyuchenko, O. Maslyuchenko, V. Melnyk, *Existence of intermidiate piecewise linear and infenitely differentiable functions*, Bukovinian Math. J., **4** (3-4) (2016), 93-100.

e-mail: windchange7@gmail.com

On discontinuity points set of separately continuous functions

Volodymyr Mykhaylyuk

Yuriy Fedkovych Chernivtsi National University, Ukraine and *Jan Kochanowski University in Kielce, Poland*

For topological spaces X and Y and a mapping $f : X \to Y$ by $D(f)$ we denote the discontinuity points set of f.

We say that a subset A of the product $X \times Y$ of topological spaces X and Y is *locally projectively nowhere dense (meager)*, if for every $x \in X$ there exists a neighborhood U of x in X such that the set $A \bigcap U$ has nowhere dense (meager) projections on X and Y.

Function $f : X \times Y \to \mathbb{R}$ is called *separately continuous*, if it continuous with respect to each variable.

Theorem 1. *Let X and Y be metrizable spaces and $E \subseteq X \times Y$. Then there exists a separately continuous function $f : X \times Y \to \mathbb{R}$ with $D(f) = E$ if and only if the set E is the union of a sequence of a F_σ-sets which are locally projectively meager.*

Theorem 2. *Let $X = \prod\limits_{s \in S} X_s$ and $Y = \prod\limits_{t \in T} Y_t$ be topological products of families of metrizable separable spaces X_s and Y_t respectively. Then a set $E \subseteq X \times Y$ is the discontinuiuty points set for a separately continuous function $f : X \times Y \to \mathbb{R}$ if and only if there exist at most countable sets $S_0 \subseteq S$ and $T_0 \subseteq T$ and projectively meager F_σ-set E_0 in a product $X_0 \times Y_0$, where $X_0 = \prod\limits_{s \in S_0} X_s$ and $Y_0 = \prod\limits_{t \in T_0} Y_t$, such that $E = \mathrm{pr}^{-1}(E_0)$ where $\mathrm{pr} : X \times Y \to X_0 \times Y_0$ is the natural projection for which $\mathrm{pr}(x, y) = (x|_{S_0}, y|_{T_0})$.*

Theorem 3. *Let X, Y be Čech complete spaces, $(E_n)_{n=1}^\infty$ be a sequence of separable compact perfect projectively nowhere dense G_δ-sets E_n in $X \times Y$ and $E = \bigcup\limits_{n=1}^\infty E_n$. Then there exists a separately continuous function $f : X \times Y \to \mathbb{R}$ such that $D(f) = E$.*

Theorem 4. *Let X, Y be Čech complete spaces, $(A_n)_{n=1}^\infty$, $(B_n)_{n=1}^\infty$ be sequences of nowhere dense compact G_δ-sets A_n and B_n in X and Y respectively, $A = \bigcup\limits_{n=1}^\infty A_n$ and $B = \bigcup\limits_{n=1}^\infty B_n$. Then there exists a separately continuous function $f : X \times Y \to \mathbb{R}$ such that $\mathrm{pr}_X D(f) = A$ and $\mathrm{pr}_Y D(f) = B$.*

Example 1. There exist Eberlein compacts X and Y and nowhere dense zero sets A and B in X and Y respectively such that $D(f) \neq A \times B$ for every separately continuous function $f : X \times Y \to \mathbb{R}$.

Example 2 (CH). There exist separable Valdivia compacts X and Y, nowhere dense separable zero sets E and F in X and Y respectively such that $D(f) \neq E \times F$ for every separately continuous function $f : X \times Y \to \mathbb{R}$.

Question 1. To characterize of discontinuity points sets of separately continuous functions on the product of two Eberlein compacts.

Question 2. Let X be a metrizable compact, Y be an Eberlein compact and E be a projectively nowhere dense zero set in the product $X \times Y$. Does there exist a separately continuous function $f : X \times Y \to \mathbb{R}$ with $D(f)$=E?

References

[1] Maslyuchenko V., Mykhaylyuk V. *Separately continuous functions of many variables on the product of spaces which are the products of metrizable factors* Mat. Visnyk NTSh., **1**, 2 (2004), 77-84 (in Ukrainian).

[2] Mykhaylyuk V. *The discontinuity points set of separately continuous functions on the products of compacts*, Methods of Func. Anal. and Top., **13**, 3 (2007), 284-295.

e-mail: vmykhaylyuk@ukr.net

Dimensions of hyperspaces and non-additive measures

Oleh Nykyforchyn

Vasyl Stefanyk Precarpatian National University, Ukraine

e-mail: oleh.nyk@gmail.com

Topological invariants of BPS states and Higgs coupling measurements at the LHC

Tetiana Obikhod

Institute of Nuclear Research NAS of Ukraine

Since only the Higgs boson was discovered among the theoretically predicted particles, the further development of physics is associated with the study of the properties of this particle. Standard Model (SM) has a few pressing questions:

• the baryon asymmetry of the Universe;

• the large mass hierarchy;
• the prevalence of the Dark Matter in the Universe;
which call for physics beyond the SM.

The most promising methods for studying of new physics are the experiments connected with the properties of the Higgs boson coupling [1, 2]. As these experiments assume the formation of new particles beyond SM, there is a need for a new theory. Mathematical calculations of possible observables in the framework of such theories would be a serious help for modern physics.

One of such methods is to consider vector bundles with a four-dimensional base and a fiber – vector space with the structure group SU(5). As the problem of Higgs selfcoupling is connected with the vacuum stability, it would be appropriate to consider the instanton numbers associated with tunnel transitions between vacuum states and with different topological quantum numbers characterizing the BPS states. The BPS solitons in SUSY theories with codimension four, three, two, and one are called instantons, monopoles, vortices, and domain walls, respectively [3]. The theory has a stable domain wall interpolating between two different vacua and the energy density of the wall is nothing but the value of the central charge q, related to the integral

$$J^{\mu\nu} = \int d^3x \varepsilon^{0\mu\nu\rho} \partial_\rho W(S^2) \to [W(S^2)_{z\to+\infty} - W(S^2)_{z\to-\infty}] \ ,$$

with superpotential

$$W = \frac{\Lambda^5}{S^2} + \frac{m}{4} S^2 \ ,$$

where Λ is the scale parameter of the SU(2) gauge theory, $m = \sqrt{2\lambda} v$, S is composite operator of SU(2) group, [4]. As superpotential has a crucial role for the properties of SUSY particles [5], it is necessary to draw a conclusion about the importance of the BPS characteristics connected with topological invariants such as central charge and new particle spectra beyond the SM.

References

[1] Christoph Englert, Roman Kogler, Holger Schulz, Michael Spannowsky *Higgs coupling measurements at the LHC*, Eur. Phys. J. **C76** (2016), 393–702.

[2] Joachim Brod, Ulrich Haisch, Jure Zupan *Constraints on CP-violating Higgs couplings to the third generation*, JHEP 1311, 180 (2013).

[3] Minoru Eto, Youichi Isozumi, Muneto Nitta, Keisuke Ohashi, Kazutoshi Ohta, Norisuke Sakai, Yuji Tachikawa *Global Structure of Moduli Space for BPS Walls*, Phys.Rev. **D71** (2005), 105009.

[4] Dvali G., Shifman M. *Domain Walls in Strongly Coupled Theories*, Phys.Lett. **B396** (1997), 64-69; Erratum-ibid. **B407** (1997), 452.

[5] Malyuta Yu., Obikhod T. *Stringy approach to the Minimal Supersymmetric Standard Model*, Problems of atomic science and technology, N3 (2011). Series: Nuclear Physics Investigations (55), p.10-13.

e-mail: obikhod@kinr.kiev.ua

On narrow operators

Mikhail Popov

Yuriy Fedkovych Chernivtsi National University, Ukraine and *Pomeranian University in Slupsk, Poland*

The lectures will consist of two parts: 1) An introduction to narrow operators; 2) Some open problems on narrow operators.

e-mail: misham.popov@gmail.com

The sum of consecutive Fibonacci numbers

Skuratovskii R. V., Rudenko D. V.

MAUP, IKIT, Ukraine

The sums of the sequential Fibonacci numbers and some other sequences form new sequences. We consider such task: find all the natural values $m > 1$ such that the divisibility is fulfilled $\sum_{k=1}^{m} f_k \vdots m$, where sequence $\{f_n\}_{n=1}^{+\infty}$ is determined by the ratio $f_{n+1} = f_n + f_{n-1},\ n \geq 2$, values f_1 and f_2 - some integers that may vary. Let's consider examples of calculations.

1) Let $f_1 = 1$ and $f_2 = 1$, so we consider the Fibonacci sequence. In this case we get such values :

1, 2, 24, 48, 72, 77, 96, 120, 144, 192, 216, 240, 288, 319, 323, 336, 360, 384, 432, 480, 576, 600, 648, 672, 720, 768, 864, 960, 1008, 1080, 1104, 1152, 1200, 1224, 1296, 1320, 1344, 1368, 1440, 1517, 1536, 1680, 1728, 1800, 1920, 1944, 2016, 2064, 2160, 2208, 2304, 2352, 2400, 2448, 2592, 2640, ...

2) Let $f_1 = 2$ and $f_2 = 1$. In that case we get such values m:

1, 3, 24, 48, 72, 96, 120, 144, 192, 216, 240, 288, 336, 360, 384, 406, 432, 480, 576, 600, 648, 672, ...

Here was used the equality: $\sum_{k=1}^{n+m} f_k = f_{n+m+2} - 1$. The following statement takes place.

Theorem 11. *The sum of any m sequential Fibonacci numbers is multiple of then and only if the numbers f_m and $f_{m+1} - 1$ are multiples of m.*

Lemma 12. *Let f_{n+1}, f_{n+2}, ..., f_{n+m}, $n \geq 0$ – m arbitrary successive Fibonacci numbers. Then the next equality holds:*

$$\sum_{k=1}^{m} f_{n+k} = f_{n+m+2} - f_{n+2}$$

.

Indeed, taking into account the equality $\sum_{k=1}^{l} f_k = f_{l+2} - 1$, for all natural numbers l, we will make the following transformations:

$$\sum_{k=1}^{m} f_{n+k} = f_{n+1} + f_{n+2} + ... + f_{n+m} =$$

$$= (f_1 + f_2 + ... + f_{n+m}) - (f_1 + f_2 + ... + f_n) =$$

$$= \sum_{k=1}^{n+m} f_k - \sum_{k=1}^{n} f_k = (f_{n+m+2} - 1) - (f_{n+2} - 1) = f_{n+m+2} - f_{n+2}.$$

The criterion of divisibility of the sum of successive Fibonacci numbers by the number terms has the following form:

Theorem 13. *The sum of any m sequential Fibonacci numbers is multiple of m if and only if the numbers f_m and $f_{m+1}-1$ are multiples of m.*

We denote by $T(m)$ the period of mentioned above sums of consecutive Fibonacci numbers reduced by the modulo m.

Statement 1. *If $(\frac{5}{p}) = 1$ then the period $T(p)$ divides $p-1$ if $(\frac{5}{p}) = -1$ then $T(p)$ divides $2p+2$.*

Theorem 14. *If $m = pq$, where $p,\ q \in P$ then the period is equal to $T(p)T\left(q\right)$ viz $T\left(pq\right) = lcm(T(p), T\left(q\right))$.*

Example 15. *The period by a composite module 33 is equal to 33 according to formula $T(33) = LCM(8,\,10) = 40$, where $8 = T(3)$, $9 = T(11)$.*

Other periods are the following: $T(72) = T(9 \cdot 8) = T(9) \cdot T(8) = lcm\,(8,\ 18) = 24 \cdot 3$. The period by the composite module $m = 2 \cdot 31$ is the product $T(2) = 3$ by $T(31) = 30$. In other words $T\left(62\right) = T\left(2\right)T(31) = 30$.

Example 16. *The period by a composite module 38 is equal to according to $T(38) = T(2 \cdot 19) = lcm\,(3,\ 18) = 18$*

Theorem 17. *If $m = p^2$ then the period of the sums $T(p)$ is equal to $pT(p)$.*

Example 18. *Take into account that $T\left(7\right) = 16$, and as an example we check that $T\left(7^2\right) = T\left(7\right)7$. Indeed due calculations we obtain the following*

$$T\left(49\right) = 112 = 2^4 7 = T\left(7^2\right) = T\left(7\right)7 = (16) \cdot 7 = \left(2^4\right)7.$$

References

[1] Vorobiev N. *Fibonacci numbers.* M .: Nauka, 1978. 144 p.

[2] Stakhov A.P. *Metal proportions are the new mathematical constants of nature.* "Academy of Trinitarianism", M., El. No. 77-6567, publ.14748, 22.03.2008.

[3] Novosad M.V, Dykcha I.A. *Luke's numbers.* Scientific herald of Chernivtsi University. 2009. 446. pp. 11-15

[4] R.V. Skuratovskii, Rudenko D. V. *The sum of consecutive Fibonacci numbers* Scientific journal FMO, (2018), v.1, p. 305-310.

e-mail: ruslcomp@mail.ru

Numerical index with respect to an operator

Alicia Quero de la Rosa

Universidad de Granada, Spain

The concept of numerical index was introduced by G. Lumer in 1968 in the context of the study and the classification of operator algebras.

This is a constant of a Banach space relating the behaviour of the numerical range with that of the usual norm on the Banach algebra of all bounded linear operators on the space. Recently, Ardalani introduced new concepts of numerical range and numerical radius of one operator with respect to another one, which generalize in a natural way the classical concepts of numerical range and numerical radius. The aim of this talk is to study basic properties of these new concepts, present some examples and provide results on the stability of the numerical index with respect to an operator under some natural operations such as absolute sums or c_0, ℓ_1 and ℓ_∞-sums and on computing the numerical index with respect to an operator of some vector-valued function spaces.

References

[1] M. A. Ardalani, Numerical index with respect to an operator, *Studia Math.* **225** (2014), 165–171.

[2] M. Martín, On different definitions of numerical range, *J. Math. Anal. Appl.* **433** (2016), 877–886.

[3] V. Kadets, M. Martín, J. Merí and A. Pérez, *Spear operators between Banach spaces*, Lecture Notes in Mathematics, **2205**, Springer, Cham 2018.

e-mail: aliciaquero@ugr.es

The Bishop-Phelps-Bollobás property
and some related topics

Óscar Roldán Blay
Universitat de València, Spain

The origin of the study of norm-attaining operators between Banach Spaces is the Bishop-Phelps Theorem [2] about the density of norm-attaining functionals. In 1961, E. Bishop and R. R. Phelps proved that given any Banach Space X over the field $\mathbb{K}$ of real or complex numbers, the set of functionals over X (this is, bounded linear operators from X to $\mathbb{K}$) is dense in the topological dual of X, denoted by X^*. In 1963, J. Lindenstrauss [5] studied whether the result remained true when we consider norm-attaining operators between two Banach Spaces X and Y, instead of functionals, and gave a negative answer to the question. As of that moment, many papers have been published worldwide studying this topic.

In 1970, B. Bollobás [3] gave a refined version of Bishop-Phelps Theorem, with which one can approximate at the same time a functional and a vector in which it almost attains its norm by a norm-attaining functional and a vector in which it attains its norm. This is the starting point of the Bishop-Phelps-Bollobás property, introduced in 2008 by M. D. Acosta, R. M. Aron, D. García and M. Maestre [1]. The Bishop-Phelps-Bollobás property (the BPBp for short) is just an adaptation of Bollobás result to operators between Banach Spaces instead of functionals. Clearly, it is a stronger property than the density of norm-attaining operators, and actually, it is known that they are not equivalent. Ever since the property was introduced, there have been numerous papers written studying this topic and similar ones. In 2013, A. J. Guirao and O. Kozhushkina [4] adapted the BPBp for operators that attain their numerical radius instead of their norm, and that started another field of study in Functional Analysis. In this talk we will see some of the main results that have been found about these topics.

References

[1] M. D. Acosta, R. M. Aron, D. García and M. Maestre, *The Bishop-Phelps-Bollobás theorem for operators*, J. Funct. Anal. **254** (2008), 2780–2799.

[2] E. Bishop and R. R. Phelps, *A proof that every Banach space is subreflexive*, Bull. Amer. Math. Soc. **67** (1961), 97–98.

[3] B. Bollobás, *An extension to the theorem of Bishop and Phelps*, Bull. London Math. Soc. **2** (1970), 181–182.

[4] A. J. Guirao and O. Kozhushkina, *The Bishop-Phelps-Bollobás property for numerical radius in* $\ell_1(\mathbb{C})$, Studia Mathematica **218**, no. 1 (2013), 41–54.

[5] J. Lindenstrauss, *On operators which attain their norm*, Israel J. Math **1** (1963), 139–148.

e-mail: osrol@alumni.uv.es

An analog of the Schwarz lemma for regular homeomorphisms

Ruslan Salimov and **Mariia Stefanchuk**

Institute of Mathematics National Academy of Sciences of Ukraine

Let $\mathbb{B} = \{z \in \mathbb{C} : |z| < 1\}$ and $f\colon \mathbb{B} \to \mathbb{C}$ be a regular homeomorphism of Sobolev class $W^{1,1}_{\mathrm{loc}}$. For any $p > 1$, the quantity

$$D_p(z) = D_p(re^{i\vartheta}) = \frac{|f_\vartheta(re^{i\vartheta})|^p}{r^p J_f(re^{i\vartheta})}$$

is called p-*angular dilatation* of f at a point $z \in \mathbb{B}$, $z \neq 0$, with respect to the origin.

Let $B_r = \{z \in \mathbb{C} : |z| < r\}$, $\gamma_r = \{z \in \mathbb{C} : |z| = r\}$. For $p > 1$, denote

$$d_p(r) = \left(\frac{1}{2\pi r}\int\limits_{\gamma_r} D^{\frac{1}{p-1}}(z)|dz|\right)^{p-1}.$$

Theorem 1. *Let $f\colon \mathbb{B} \to \mathbb{B}$ be a regular homeomorphism of Sobolev class $W^{1,1}_{\mathrm{loc}}$ possessing Lusin's (N)-property, and $f(0) = 0$. Suppose that there exists k such that for $p > 2$,*

$$\liminf_{r\to 0}\left(\frac{1}{\pi r^2}\iint\limits_{B_r} D_p^{\frac{1}{p-1}}(z)\,dxdy\right)^{p-1} \leq k < \infty.$$

Then

$$\liminf_{z\to 0}\frac{|f(z)|}{|z|} \leq c_p\, k^{\frac{1}{p-2}} < \infty,$$

where c_p is a positive constant depending only on p.

Theorem 2. *Let $f\colon \mathbb{B} \to \mathbb{B}$ be a regular homeomorphism of Sobolev class $W^{1,1}_{\mathrm{loc}}$ possessing Lusin's (N)-property and normalized by $f(0) = 0$. Suppose that $p > 2$ and there exists k_0 such that*

$$k_0 = \limsup_{r\to 0} r^{p-2}\int\limits_r^1 \frac{dt}{t^{p-1}d_p(t)}.$$

Then

$$\liminf_{z\to 0}\frac{|f(z)|}{|z|} \leq (p-2)^{\frac{1}{2-p}}\, k_0^{\frac{1}{2-p}}.$$

e-mails: ruslan.salimov1@gmail.com, stefanmv43@gmail.com

A gentle introduction to forcing

Olga Sipacheva
Moscow State University, Russian Federation

e-mail: osipa@gmail.com

Borel sets in topological spaces

Jiří Spurný
Charles University in Prague, Czech Republic

We present a construction of the Borel hierarchy in general topological spaces and its relation to Baire hierarchy. We define mappings of Borel class α, prove the validity of the Lebesgue–Hausdorff–Banach characterization for them and show their relation to Baire classes of mappings on compact spaces. We prove a key theorem on invariance of Borel sets with respect to perfect mappings. The obtained results are used for studying Baire and Borel order of compact spaces. We present several examples showing some natural limits of our results in non-compact spaces. We also include applications of the obtained results to the theory of compact convex sets.

e-mail: spurny@karlin.mff.cuni.cz

The finding of the number of the nonisomorfic (n, m)-graphs

Polina Stegantseva and **A. Artemenko**
Zaporizhzhya National University, Ukraine

The simple graph G with n vertices and m edges is called (n,m)-graph. Let $T(n,m)$ is the number of the nonisomorfic (n,m)-graph. In [3] we can find $T(n,m)$ for $n=\overline{1,18}$ and in [1] we can find the sequence $A008406$ for $n=\overline{1,20}$. In [2] the formula for the generating function of the number of the simple graphs has been obtained. This work deals with the simple way of the calculation of the some number $T(n,m)$ for $n=\overline{21,26}$.

Some properties:

1) Let is the set of all nonisomorfic (n,m)-graphs. Then the subset, which is equivalent to M, exists in the set of all nonisomorfic $(n+1,m)$-graphs. Arbitrary graph of the subset differs from any graph of the set by one isolated vertex.

2) If the graph with the vector of the degrees $(1,1,,1)$ exists in the set of all nonisomorfic (n,m)-graphs, then $n=2m$.

3) Let k is the number of all nonisomorfic $(2m,m)$-graphs. Then the number of all nonisomorfic (n,m)-graphs equals k for all $n>2m$.

Definition 1. Let $(s_1,s_2,...,s_n)$ is the vector of the degrees of (n,m)-graph G. Let for any i takes place $s_i\geq 2$ and (v_i,v_j) is one of the edges. We add new isolated vertex v_{n+1}. We move off the edge (v_i,v_j) and we add the edge (v_j,v_{n+1}). Then the degree of the vertex v_i is reduced by 1 and the degree of the new vertex is 1. In this case we call the transformation of the graph G P-transformation.

Using P-transformation we can transform $(2m-k,m)$-graph for $k>0$ into $(2m,m)$-graph. The following formulas have been proved:

$$T(2m,m)=T(2m,1)+T(2m-1,m),m>1$$

$$T(2m,m)=T(2m,2)+T(2m-2,m),m>2$$

$$T(2m,m)=T(2m,3)+T(2m-3,m),m>3$$

$$T(2m,m)=T(2m,4)+T(2m-4,m)+1,m>5$$

$$T(2m,m)=T(2m,5)+T(2m-5,m)+4,m>7$$

$$T(2m,m)=T(2m,6)+T(2m-6,m)+4,m>9$$

We calculate $T(21,11)$, $T(22,11)$, $T(21,12)$, $T(22,12)$, $T(23,12)$, $T(24,12)$, $T(21,13)$, $T(22,13)$, $T(23,13)$, $T(24,13)$, $T(25,13)$, $T(26,13)$.

References

[1] https://oeis.org

[2] Bedratyuk L. and Bedratyuk A. *A new formula for the generating function of the numbers of simple graphs*, arXiv:1512.06355 [math.CO], 2015.

[3] Harari F., Palmer E. *Perechislenie grafov* , 1977.

e-mails: stegpol@gmail.com, 1997.engel@mail.ru

GJMS operator on Einstein Riemannian manifold

Kamel Tahri

High School of management

Abou Bekr Belkaid University of Tlemcen, Algeria

e-mail: Tahri_kamel@yahoo.fr

On topological aspects of the digital image segmentation

Iryna Yurchuk and **Maksym Pasichnyk**

National Aviation University, Ukraine

Introduction. Nowdays the data mining needs new effective methods of data proccessing caused by the growth of data flow. The image segmentation is usufull part of a machine vision, an object detection, the recognition tasks and etc. The most of image segmentation methods are based on statistics methods that in fact lead to some disadvantages. For example, K-means algorithm requires to know the quantity of clusters in advance that restricts its application.

The authors propose the image segmentation algorithm based on the persistent homologies as the effective mode of topological data analysis.

Background. Terms of computational topology are formulated, see [1, 3, 4]. Given $S \subset Y$ and $\epsilon \in \mathbb{R}$, let $G_\epsilon = \{S, E_\epsilon\}$ be ϵ-neighborhood graph on S, where $E_\epsilon = \{(u, v) : d(u, v) \leq \epsilon, u \neq$

$v \in S\}$. *A clique* in a graph is the subset of vertices that induces a complete subgraph. The clique complex has the maximal cliques of a graph as its maximal simplices.

The Vietoris-Rips complex C_ϵ is the clique complex of ϵ-neighborhood graph.

A filtration of a space X is a nested sequence of subspaces: $\emptyset \subseteq X_1 \subseteq \ldots \subseteq X_n = X$.

For $\epsilon \leq \epsilon'$, it is true that $C_\epsilon \subseteq C_{\epsilon'}$. The set $\{C_{\epsilon_i}\}_{i=1}^k$ of Vietoris-Rips complexes is the filtration for any finite set $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_k\}$, where $\epsilon_i < \epsilon_j$, $i < j$.

Let $\mathrm{H}_p^i = \mathrm{H}_p(C_{\epsilon_i})$, where H_p be $p-$th homology, and $f_p^{i,j} : \mathrm{H}_p^i \to \mathrm{H}_p^j$, $i < j$, be a map.

The $p-$th *persistent homology* $\mathrm{H}_p^{i,j}$ is $\mathrm{Im} f_p^{i,j}$ for $0 \leq i < j \leq k+1$. On other words, $\mathrm{H}_p^{i,j} = Z_p^i \,/ (B_p^j \cap Z_p^i)$, where Z_p^i is $p-$cycles of C_{ϵ_i} and B_p^j is $p-$boundaries of C_{ϵ_j}. There is a method of their calculation based on the matrices algebra, the persistence barcode and the persistence diagrams (see [4]).

It's known that $\beta_0 = \mathrm{rank}\ \mathrm{H}_0^{i,j}$ is the amount of connected components of the space. For the digital image segmentation, it is the same as the quantity of clusters.

Practice. Let consider a digital image D as a set of points M in R^5. Denote by L the length of D and W the width of D. Every pixel P has two parameters of the plane location (x and y) and three color components (for example, RGB). The digital image segmentation algorithm based on a persistent homology is following:

1. Let $P(x, y, r, g, b)$ be a pixel of D. Then the coordinates of every P are normalized and the following is obtained: $x' = \dfrac{x}{\max\{L, W\}}$, $y' = \dfrac{y}{\max\{L, W\}}$, $r' = \dfrac{r}{255}$, $g' = \dfrac{g}{255}$ and $b' = \dfrac{b}{255}$, where x',y',r',g',$b' \in [0; 1]$. The set M is transformed into the new set M' in R^5;

2. Fix a finite set $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_k\}$ such that $\epsilon_i < \epsilon_j$ for $i < j$. Construct the filtration of Vietoris-Rips complexes $\{C_{\epsilon_i}\}_{i=1}^k$ on the even grid of the set M';

3. Construct the matrices of persistent homologies of $\{C_{\epsilon_i}\}_{i=1}^k$;

4. Calculate the rank of Smith normal form of persistent homologies matrices. This number is the quantity of segmentation clusters.

The image segmentation algorithm based on the persistent homologies is implemented in $C\sharp$ $(.NET4.5)$ and the results are compared with the K-means algorithm.

Conclusions. After testing on real images it becomes obvious that the digital image segmentation algorithm based on the persistent homologies is more effective than K-means algorithm and not sensitive even to high noise levels.

References

[1] Carlsson G. *Topology and data*, Bull.Amer.Math.Soc. **46 (2)** (2009), 255–308.

[2] Dumas J.-G., Heckenbach F., SaundersvB. D., Welker V. *Computing simplicial homology based on efficient Smith normal form algorithms*, Algebra, Geometry, and Software Systems (2003), 177-207.

[3] Edelsbrunner H., Letscher D., Zomorodian A.*Topological persistence and simplification*, Disc.Comput.Geom. **28 (4)** (2002), 511–533.

[4] Zomorodian A., Carlsson G. *Computing persistent homology*, Disc.Comput.Geom. **33 (2)** (2005), 249–274.

e-mails: i.a.yurchuk@gmail.com, scrumbell@gmail.com

Non-expansive bijections and EC-plasticity of the unit balls

Olesia Zavarzina

V.N. Karazin Kharkiv National University, Ukraine

A metric space E is said to be expand-contract plastic (or briefly an EC-space) if every non-expansive bijection $F\colon E \to E$ is an isometry. The EC-plasticity of totaly bounded metric spaces was established in [4, Theorem 1.1]. In particular, the unit ball or even every bounded set of any finite-dimensional Banach space is EC-plastic. However, the question about plasticity of the unit ball of an arbitrary Banach space is open. In [1, Theorem 2.6] the mentioned property is proved for the unit balls of strictly convex Banach spaces. So, the unit balls of the spaces L_p with $1 < p < \infty$ as well as of all Hilbert spaces are plastic. On the other hand, there is an example of bounded closed convex set in an infinite-dimensional Hilbert space that is not EC-space [1, Example 2.7]. As we can see, the question about plasticity in infinite-dimensional spaces is much more difficult. We answered this question in positive for the unit ball of the space ℓ_1, which is not strictly convex.

Theorem 1 ([2, Theorem 1]). *The unit ball of ℓ_1, is an EC-space.*

Another interesting question connected with the previous one is about unit balls of two different Banach spaces X and Y. If one consider a non-expansive bijection $F\colon B_X \to B_Y$, for which spaces X and Y the mapping F appears to be an isometry? In this field we received some generalization. For cases when the unit ball of the space Y is known to be EC-plastic, we get that X may be an arbitrary space. To be more exact, the following theorems hold.

Theorem 2 ([5, Theorem 3.1]). *Let $F\colon B_X \to B_Y$ be a bijective non-expansive map. If Y is strictly convex, then F is an isometry.*

Theorem 3 ([5, Theorem 3.5]). *Let $F\colon B_X \to B_{\ell_1}$ be a bijective non-expansive map. Then F is an isometry.*

Theorem 4 ([5, Theorem 3.8]). *Let Y be a finite-dimensional Banach space, $F\colon B_X \to B_Y$ be a bijective non-expansive map. Then F is an isometry.*

We have also done one more step in solving the problem with the unit balls of two different spaces.

Theorem 5 ([3, Theorem 3.1]). *Let X be a Banach space, $Z_i, i \in I$ be a fixed collection of strictly convex Banach spaces, Z be the ℓ_1-sum of the collection $Z_i, i \in I$, and $F\colon B_X \to B_Z$ be a non-expansive bijection. Then F is an isometry.*

As a consequence, we get the EC- plasticity of the unit ball of the space Z.

References

[1] B. Cascales, V. Kadets, J. Orihuela, E.J. Wingler. Plasticity of the unit ball of a strictly convex Banach space. *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas*, 110 (2) : 723–727, 2016.

[2] V. Kadets, O. Zavarzina. Plasticity of the unit ball of ℓ_1, *Visn. Hark. nac. univ. im. V.N. Karazina, Ser.: Mat. prikl. mat. meh.* 83 : 4–9, 2017.

[3] V. Kadets, O. Zavarzina. Non-expansive bijections to the unit ball of ℓ_1-sum of strictly convex Banach spaces, *Bulletin of the Australian Mathematical Society*, 97(2) : 285–292, 2018.

[4] S.A. Naimpally, Z. Piotrowski, E.J. Wingler. Plasticity in metric spaces. *J. Math. Anal. Appl.*, 313 : 38–48, 2006.

[5] O. Zavarzina. Non-expansive bijections between unit balls of Banach spaces. *Ann. Funct. Anal.*, 9 (2): 271–281, 2018.

e-mail: olesia.zavarzina@yahoo.com

Basics of proper forcing, with emphases on posets consisting of trees

Lyubomyr Zdomskyy

Vienna University of Technology, Austria

e-mail: lzdomsky@gmail.com

Absoluteness Theorems

Szymon Żeberski
Wroclaw University of Science and Technology, Poland

e-mail: szymon.zeberski@pwr.edu.pl