The 13th Summer School

ANALYSIS, TOPOLOGY
and APPLICATIONS

July 29 – August 11, 2018

Vyzhnytsya, Chernivtsi Region, Ukraine

BOOK OF ABSTRACTS
The 13th Summer School “Analysis, Topology and Applications”,
July 29 – August 11, 2018
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Book Of Abstracts

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For a locally convex vector space $(V, \tau)$ there exists a finest locally convex vector space topology $\mu$ such that the topological dual spaces $(V, \tau)'$ and $(V, \mu)'$ coincide algebraically. This topology is called Mackey topology. If $(V, \tau)$ is a metrizable locally convex vector space, then $\tau$ is the Mackey topology.

In 1995 Chasco, Martín Peinador and Tarieladze asked the following question: Given a locally quasi–convex group $(G, \tau)$, does there exist a finest locally quasi–convex group topology $\mu$ on $G$ such that the character groups $(G, \tau)^\wedge$ and $(G, \mu)^\wedge$ coincide?

Two locally quasi–convex group topologies $\tau$ and $\tau'$ on an abelian group $G$ are named compatible if the character groups $(G, \tau)^\wedge$ and $(G, \tau')^\wedge$ coincide algebraically. The set of group topologies, which are compatible with a given group topology, forms a partially ordered set with the weak topology as bottom element. In case there exists a top element, it is called the Mackey topology.

It was shown in the above mentioned paper that every locally compact group topology on an abelian group is the Mackey topology.

In the talk we first examine general result concerning compatible group topologies. Afterwards, we study compatible group topologies on LCA groups. (Surprisingly, the cases of $\mathbb{R}$, $\mathbb{Z}$ and the Prufer groups $\mathbb{Z}(p^{\infty})$ are the most difficult ones.)

Then we give a survey on groups where the given topology is / is not the Mackey topology. Finally we present a locally quasi–convex group which does not admit a Mackey topology.

References


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On complete topologized semilattices

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and

Ivan Franko National University of Lviv, Ukraine

In the lectures with shall survey some properties of complete topologized semilattices.

A topologized semilattice is a semilattice endowed with a topology. A topologized semilattice is complete if each chain $C \subset X$ has $\inf C$ and $\sup C$ that belong to the closure of the chain $C$ in $X$.

We shall prove that each complete topologized semilattice $X$ is closed in any functionally Hausdorff semitopological semilattice that contains $X$. Also we shall prove that the partial order in a complete functionally Hausdorff semitopological semilattice is a closed subset of $X \times X$.

More details can be found in the following preprints.

References


T. Banakh, S. Bardyla, *On images of complete topologized subsemilattices in sequential semitopological semilattices.*

T. Banakh, S. Bardyla, *The interplay between weak topologies on topological semilattices.*

T. Banakh, S. Bardyla, *Characterizing chain-compact and chain-finite topological semilattices.*

T. Banakh, S. Bardyla, *Completeness and absolute H-closedness of topological semilattices.*

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**Weak topologies on topologized semilattices**

**Serhii Bardyla**

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We will discuss an interplay between weak topologies on topologized semilattices.

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**Curvature properties of LCK-manifolds and their submanifolds**

**Yevhen Cherevko** and **Olena Chepurna**

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**Definition 1.** A Hermitian manifold \((M^{2m}, J, g)\) is called a locally conformal Kähler manifold (LCK - manifold) if there is an open cover \(\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}\) of \(M^{2m}\) and a family \(\{\sigma_\alpha\}_{\alpha \in \Lambda}\) of \(C^\infty\) functions \(\sigma_\alpha : U_\alpha \to \mathbb{R}\) so that each local metric

\[
\hat{g}_\alpha = e^{-2\sigma_\alpha} g|_{U_\alpha}
\]
is Kählerian. An LCK - manifold is endowed with some form \( \omega \), so called a Lee form which can be calculated as [1]

\[
\omega = \frac{1}{m-1} \delta \Omega \circ J,
\]

The form should be closed:

\[
d\omega = 0.
\]

There is A. Gray’s[2] classification of Hermitian manifolds : \( M^{2m} \in \mathcal{L}_i \) if the curvature tensor \( R \) of \( M^{2m} \) satisfies the identity \((i)\) where

1. \( R(X, Y, Z, W) = R(X, Y, JZ, JW) \)
3. \( R(X, Y, Z, W) = R(JX, JY, JZ, JW) \)

Vaisman[3] have proved that if \( M^{2m} \) is LCK–manifold, then \( \mathcal{L}_2 = \mathcal{L}_3 \).

It’s worth for noting that the classification isn’t a complete one. For example, the generalized Hopf manifolds are included in neither class. We study immersion \( \Psi : \bar{M}^{2m-1} \rightarrow M^{2m} \) which is locally represented by functions

\[
x^\alpha = x^\alpha(y^1, \ldots, y^{2m-1}),
\]

where \( \alpha = 1, \ldots, 2m \), \( y^i, i = 1, \ldots, 2m - 1 \) is a local coordinate system in \( \bar{M}^{2p} \). We put here

\[
B^\alpha_i = \partial_i x^\alpha.
\]

We obtained the theorems.

**Theorem 1.** If a hypersurface \( \bar{M}^{2m-1} \) of a LCK-manifold \( M^{2m} \) is an integral manifold of the distribution defined by the equation

\[
\omega = 0
\]

where \( \omega \) is Lee form of the LCK-manifold \( M^{2m} \) that satisfies the condition

\[
\nabla_X \omega(Y) = \frac{1}{4} ||\omega||^2 g(X, Y) - \frac{1}{2} \omega(X)\omega(Y),
\]
i.e. $M^{2m} \in \mathcal{L}_1$, then induced by the immersion the almost contact structure

1) $f^j_i = J_\beta^\alpha B^j_\alpha B^\beta_i$;
2) $\eta_k = \frac{1}{||\omega||} B^\beta_k J_\beta^\alpha \omega_\alpha$;
3) $\xi^k = -\frac{1}{||\omega||} B^k_\beta J_\beta^\alpha \omega_\alpha$.

is a c-Sasakian structure, where $c = \frac{1}{2}||\omega||$. Moreover, the manifold $\overline{M}^{2m-1}$ is a totally umbilical hypersurface of $M^{2m}$.

**Theorem 2.** If a hypersurface $\overline{M}^{2m-1}$ of an LCK-manifold $M^{2m}$ is an integral manifold of the distribution defined by the equation

$$\omega = 0$$

where $\omega$ is Lee form of the LCK-manifold $M^{2m}$ that satisfies the condition

$$\nabla_X \omega(Y) = \frac{1}{2} ||\omega||^2 g(X,Y),$$

then induced by the immersion the almost contact structure

1) $f^j_i = J_\beta^\alpha B^j_\alpha B^\beta_i$;
2) $\eta_k = \frac{1}{||\omega||} B^\beta_k J_\beta^\alpha \omega_\alpha$;
3) $\xi^k = -\frac{1}{||\omega||} B^k_\beta J_\beta^\alpha \omega_\alpha$.

is the cosymplectic structure i.e. a normal almost contact metric structure for which the conditions

1) $d\eta = 0$, 2) $d\bar{\Omega} = 0$, 3) $n^k_{ij} = 0$;

are satisfied. Moreover, $\overline{M}^{2m-1}$ is a totally umbilical hypersurface of $M^{2m}$.

Unfortunately, the LCK–manifolds whose Lee forms satisfy the equation $\nabla_X \omega(Y) = \frac{1}{2} ||\omega||^2 g(X,Y)$ are included in neither class $\mathcal{L}_i$.

**References**

The study of the density of norm attaining linear operators has its root in the classical Bishop-Phelps theorem which states that the set $\text{NA}(X, \mathbb{R})$ of those linear functionals $T$ from a Banach space $X$ to $\mathbb{R}$ which attain their norm (i.e. there exists $x$ in the unit sphere of $X$ such that $|T(x)| = \|T\|$) is dense in the dual space of $X$. The aim of this talk is to study the set $\text{SNA}(M, Y)$ of those Lipschitz operators from a metric space $M$ to a Banach space $Y$ which attain their Lipschitz norm, as a nonlinear generalisation of the norm attaining linear operators theory. We show some metric spaces $M$ such that $\text{SNA}(M, \mathbb{R})$ is not norm dense and give conditions which are sufficient to provide the norm density of $\text{SNA}(M, Y)$ in the space of all Lipschitz operators from $M$ to any Banach space $Y$, studying the relationship between them.

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### About one orthogonal trigonometric Schauder basis for the space $C(T^2)$

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Let $\mathbb{T}^2 \cong [-\pi, \pi)^2$ be the 2-dimensional torus and $C(\mathbb{T}^2)$ be the space of $2\pi$-periodic in each variable continuous on $\mathbb{R}^2$ functions. Our main result is the construction of an orthogonal trigonometric Schauder basis for the space $C(\mathbb{T}^2)$ [1]. Further for this basis we use notation $\{t_k\}_{k \in \mathbb{N}}$.

Our results generalize the one-dimensional construction that is based on the kernel of de la Vallée Poussin [2]. To construct this basis we use ideas of a dyadic anisotropic periodic multiresolution analysis (PMRA) and corresponding wavelet spaces that were developed in [3] and [4]. The multiresolution analysis is formed using the sequence of only rotation matrices. The polynomial degree is considered in terms of the $l_1$- and $l_\infty$-norms.

For a function $f \in C(\mathbb{T}^2)$ and $\mu \in \mathbb{N}$ we define the operator

$$S_\mu f = \sum_{k=1}^{\mu} \langle f, t_k \rangle t_k,$$

where $\langle f, t_k \rangle$ are the Fourier coefficients with respect to the basis $\{t_k\}_{k \in \mathbb{N}}$.

The focus of attention is the estimation of the norm $\|S_\mu\|_{C(\mathbb{T}^2) \to C(\mathbb{T}^2)}$.

This is joint work with Vitalii Myroniuk (Institute of Mathematics NAS Ukraine) and Jürgen Prestin (Institut für Mathematik, Universität zu Lübeck).

References


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CATEGORICAL RESOLUTIONS OF SINGULAR CURVES

Yuriy Drozd

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It is a survey of a joint work with Igor Burban.

We construct a categorical resolution of a singular curve using a certain sheaf of orders of finite global dimension. Actually, we show that the derived category of coherent sheaves over this sheaf of orders provides a recollement of the derived category of coherent sheaves over the original curve. In rational case we also show that the resulting derived category is equivalent to the derived category of modules over a quasi-hereditary finite dimensional algebra.

I am also going to explain why the “usual” resolution is not enough, at least from the homological point of view.

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THE AMENABILITY TO ALGEBRAIC AND ANALYTICAL PERSPECTIVE

Rachid El Harti

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The talk is a synthesis on my contributions that I have done in several institutes on the subject of the amenability to algebraic and analytical perspective since 2003 and is based on the following papers.

References


The semisimplicity of amenable operator algebras. Archiv der Mathe-
matik August 2013, Volume 101, Issue 2, pp 129-133 (with Paulo Pinto).

Amenable Cross product Banach algebras associated with a class of
dynamical systems. Integral Equation and Operator Theory. 2017 (with
Paulo Pinto and Marcel de Jeu).

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Can One Hear the Shape of a Group?
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The monoid of monotone injective partial selfmaps of
the poset \((\mathbb{N}^3, \leq)\) with cofinite domains and images
Oleg Gutik and Olha Krokhmalna
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We shall follow the terminology of [1].

Let \(\mathbb{N}\) be the set of positive integers with the usual linear order
\(\leq\) and \(n\) be an arbitrary positive integer greater then or equal 2. On
the Cartesian power \(\mathbb{N}^n = \mathbb{N} \times \cdots \times \mathbb{N}\) \(n\)-times
we define the product partial
order, i.e.,

\[(i_1, \ldots, i_n) \leq (j_1, \ldots, j_n) \iff (i_k \leq j_k) \quad \forall k = 1, \ldots, n.\]

The set \(\mathbb{N}^n\) with this partial order will be denoted by \(\mathbb{N}^n_{\leq}\).

For an arbitrary positive integer \(n \geq 2\) by \(\mathcal{PO}_\infty(\mathbb{N}^n_{\leq})\) we denote the
semigroup of injective partial monotone selfmaps of \(\mathbb{N}^n_{\leq}\) with cofinite
domains and images.

We discuss on the semigroup of injective partial monotone self-
maps of \(\mathbb{N}^n_{\leq}\) with cofinite domains and images. We show that the
group of units of the semigroup \(\mathcal{PO}_\infty(\mathbb{N}^n_{\leq})\) is isomorphic to the group
of permutations of an \( n \)-elements set \( \mathcal{I}_n \) and describe the subgroup of idempotents of \( \mathcal{PO}_\infty(N^n) \). In the case \( n = 3 \) we describe the property of elements of the semigroup \( \mathcal{PO}_\infty(N^3) \) as partial bijections of the poset \( N^3 \) and Green’s relations on the semigroup \( \mathcal{PO}_\infty(N^3) \). Also the Witt property of the semigroup \( \mathcal{PO}_\infty(N^3) \) is discussed.

References


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**ON OLD AND NEW CLASSES OF FEEBLY COMPACT SPACES**

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In general topology are often investigated different classes of compact-like spaces and relations between them, see, for instance, basic [3, Chap. 3] and general works [2], [5], [7], [6], [4]. We consider the present paper as a next small step in this quest.

Namely, in order to refine the stratification of compact-like spaces, we introduce three new classes of pracompact spaces, consider their basic properties and relations with other compact-like spaces. Our main motivation to introduce these spaces is possible applications in topological algebra.

Relations between different classes of compact-like spaces are also well-studied. Some of them are presented at Diagram 3 in [5, p.17], at Diagram 1 in [1, p. 58] (for Tychonoff spaces), and at Diagram 3.6 in [6, p. 611]. We also present a big diagram.
References


Fractal dimensions for non-additive measures

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We follow the terminology and notation of [1] and denote by \( \exp X \) the set of all non-empty closed subsets of a compactum \( X \). We call a function \( c : \exp X \cup \{\emptyset\} \to I \) a capacity on a compactum \( X \) if the three following properties hold for all subsets \( F, G \subseteq X \):

1. \( c(\emptyset) = 0 \);
2. if \( F \subseteq G \), then \( c(F) \leq c(G) \) (monotonicity);
3. if \( c(F) < a \), then there is an open subset \( U \supseteq F \) such that for all \( G \subseteq U \) the inequality \( c(G) < a \) is valid (upper semicontinuity).

If, additionally, \( c(X) = 1 \) (or \( c(X) \leq 1 \)) holds, then the capacity is called normalized (resp. subnormalized). We denote by \( M X \) and \( MX \) the sets of all normalized and of all subnormalized capacities respectively.
We consider the metric $\hat{d}$ on the set $\bar{M} X$ of subnormalized capacities on a metric compactum $(X, d)$:

$$\hat{d}(c, c') = \inf \{ \varepsilon > 0 \mid c(\bar{O}_\varepsilon(F)) + \varepsilon \geq c'(F), c'(\bar{O}_\varepsilon(F)) + \varepsilon \geq c(F), \forall F \subset X \},$$

here $\bar{O}_\varepsilon(F)$ is the closed $\varepsilon$-neighborhood of a subset $F \subset X$. This metric determines a compact topology on $\bar{M} X [1]$. 

We investigate the problem approximation of arbitrary capacity with regular additive measures on a finite subspace. The article [2] proposes an algorithm which enables to find an additive measure $m$ on subspace $X_0 = \{ x_1, x_2, \ldots, x_n \} \subset X$ that is (almost) the closest to capacity $c \in \bar{M} X$ with respect to the distance $\hat{d}$. The presented algorithm is convenient for programmatic implementation but it requires previously calculated values of a capacity for all $2^{\text{cardinality of the space}}$ subsets, which is not appropriate even for $\geq 40$ points. Hence, to handle subspaces of greater cardinality, we need investigate dimensional characteristics of capacities.

For a capacity $c \in \bar{M} X$, among all such $X_0 \subset X$ that $c(A) = c(A \cap X_0)$ for all $A \subset X$, there is a smallest set, which is called the support of $c$ and denoted by $\text{supp } c$.

For a capacity $c \in \bar{M} X$ and a number $\varepsilon \geq 0$, a collection $\mathcal{F}$ of closed sets in $X$ is called an $\varepsilon$-foundation of $c$ if, for each non-empty $A \subset X$, there are $F_1, F_2, \ldots, F_n \in \mathcal{F}$, $n \in \mathbb{N}$, such that $F_i \cap A \neq \emptyset$, for all $1 \leq i \leq n$, and $c(F_1 \cup F_2 \cup \cdots \cup F_n) \geq c(A) - \varepsilon$.

An $\varepsilon$-foundation $\mathcal{F}$ of $c$ is called a $\delta$-$\varepsilon$-foundation, for $\varepsilon \geq 0$, $\delta > 0$, if $\text{diam } F \leq \delta$ for all $F \in \mathcal{F}$.

For $c \in \bar{M} X$, $s \geq 0$, $\varepsilon \geq 0$, $\delta > 0$, we denote:

$$N_{\delta, \varepsilon}(c) = \min \{ |\mathcal{F}| \mid \mathcal{F} \text{ is a } \delta$-$\varepsilon$-foundation of $c \},$$

$$\mathcal{H}^s_{\delta, \varepsilon}(c) = \inf \{ \sum (\text{diam } F)^s \mid F \in \mathcal{F} \} \mid \mathcal{F} \text{ is a } \delta$-$\varepsilon$-foundation of $c \}.$$ 

We define the weak upper box dimension, the weak lower box di-
mension, and the weak Hausdorff dimension of capacity $c$:

$$\dim_{WB} c = \lim_{\varepsilon \to 0} \dim_{\varepsilon B} c,$$
$$\dim_{WB} c = \lim_{\varepsilon \to 0} \dim_{\varepsilon B} c,$$
$$\dim_{WH} c = \lim_{\varepsilon \to 0} \dim_{\varepsilon H} c,$$

where

$$\dim_{\varepsilon B} c = \lim_{\delta \to 0} \frac{\ln N_{\delta,\varepsilon}(c)}{-\ln \delta}, \quad \dim_{\varepsilon H} c = \sup \{ s \geq 0 \mid H^s(\varepsilon c) = \infty \} = \inf \{ s \geq 0 \mid H^s(\varepsilon c) = 0 \}.$$ 

**Proposition 1.** There is a normalized capacity $c$ on $I^2$ such that $\dim_{WB} c = \dim_{WB} c < \dim_{B \supp c} = \dim_{B \supp c}$.

In general, all these dimensions for a capacity $c \in MX$ are different (some may be equal) and do not exceed the corresponding classical dimensions [3] of support $\supp c$:

**Proposition 2.** $\dim_{WH} c \leq \dim_{WB} c \leq \dim_{WB} c$,
$$\dim_{WH} c \leq \dim_{B \supp c}, \quad \dim_{WB} c \leq \dim_{B \supp c},$$
$$\dim_{WB} c \leq \dim_{B \supp c} for all c \in MX.$$

**References**


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Measurable selectors of multifunctions in non-separable spaces

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Let $X$ be a real Banach space, and $(\Omega, \Sigma, \mu)$ be a complete finite measure space. A multifunction is a map $F : \Omega \to 2^X \setminus \{\emptyset\}$, a selector of $F$ is a function $f : \Omega \to X$ with $f(t) \in F(t)$ for all $t \in \Omega$.

**Theorem 1** (Kuratowski and Ryll-Nardzewski, 1965). Let $X$ be a separable Banach space, and let $F : \Omega \to 2^X \setminus \{\emptyset\}$ be a multifunction that takes closed values and satisfies the following Effros measurability condition: for each open set $U \subset X$ the subset $\{t \in \Omega : F(t) \cap C \neq \emptyset\}$ belongs to $\Sigma$. Then $F$ admits a Borel measurable selector.

The Kuratowski and Ryll-Nardzewski theorem enables to define an integral of multifunctions through integrals of its selectors, and leads to a number of types of integrals (like strong, weak or Pettis ones) for multifunctions. There are definitions of multifunction integral, that are not based on selectors, but in order to get good properties of that types of integral one also needs some selection theorems. By this reason the multifunctions integration theory deals mostly with multifunctions whose values are subsets of a separable Banach space. Some years ago we started with Bernardo Cascales and José Rodríguez a joint research project whose goal was to build in non-separable spaces an acceptably good theory of integration of multifunctions.

A multi-function $F : \Omega \to 2^X$ is said to be scalarly measurable, if for every $x^* \in X^*$ the function $t \mapsto \sup x^*(F(t))$ is measurable. In particular a single valued function $f : \Omega \to X$ is scalarly measurable if the composition $x^* \circ f$ is measurable for every $x^* \in X^*$. I am going to speak about our results about weak types of integrability for multifunctions and scalarly measurable selectors. In particular, the following theorem will be demonstrated.

**Theorem 2** (Cascales, Kadets, Rodríguez, 2010). If all the values of a scalarly measurable multifunction $F$ are weakly compact, then $F$ admits a scalarly measurable selector.
I strongly believe that the above cited theorem because of its generality may find applications in those areas of applied mathematics where the Kuratowski and Ryll-Nardzewski theorem is often used, and only my ignorance in applied mathematics does not permit me to find such applications.

References


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**Fragmentability and Related Properties**

**Olena Karlova**

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Let $X$ be a topological space and $(Y,d)$ be a metric space. A map $f : X \to Y$ is called $\varepsilon$-fragmented for some $\varepsilon > 0$ if for every closed nonempty set $F \subseteq X$ there exists a nonempty relatively open set $U \subseteq F$ such that $\text{diam} f(U) < \varepsilon$. If $f$ is $\varepsilon$-fragmented for every $\varepsilon > 0$, then it is called fragmented.

Let $\mathcal{U} = (U_\xi : \xi \in [0,\alpha])$ be a transfinite sequence of subsets of a topological space $X$. We define $\mathcal{U}$ to be regular in $X$, if

(a) each $U_\xi$ is open in $X$;

(b) $\emptyset = U_0 \subset U_1 \subset U_2 \subset \cdots \subset U_\alpha = X$;
(c) $U_\gamma = \bigcup_{\xi < \gamma} U_\xi$ for every limit ordinal $\gamma \in [0, \alpha)$.

A map $f : X \to Y$ is $\varepsilon$-fragmented iff there exists a regular sequence $\mathcal{U} = (U_\xi : \xi \in [0, \alpha])$ (which is called $\varepsilon$-associated with $f$ and is denoted by $\mathcal{U}_\varepsilon(f)$) in $X$ such that $\text{diam}_f(U_{\xi+1} \setminus U_\xi) < \varepsilon$ for all $\xi \in [0, \alpha)$.

We say that an $\varepsilon$-fragmented map $f : X \to Y$ is functionally $\varepsilon$-fragmented if $\mathcal{U}_\varepsilon(f)$ can be chosen such that every set $U_\xi$ is functionally open in $X$. Further, $f$ is functionally $\varepsilon$-countably fragmented if $\mathcal{U}_\varepsilon(f)$ can be chosen to be countable and $f$ is functionally countably fragmented if $f$ is functionally $\varepsilon$-countably fragmented for all $\varepsilon > 0$.

In the lectures we will discuss the relations between functionally countably fragmented maps and other classes of maps (Baire-one, $F_\sigma$-measurable, etc.). We will also examine the extension problem of fragmented maps.

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**Bifurcation of cycles in parabolic systems of differential equations with weak diffusion**

**Ivan Klevchuk**

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Consider the equation [1, 2]

$$\frac{\partial u}{\partial t} = i\omega_0 u + \varepsilon \left[ (\gamma + i\delta) \frac{\partial^2 u}{\partial x^2} + (\alpha + i\beta)u \right] + (d_0 + ic_0)u^2 \bar{u} \tag{1}$$

with periodic boundary condition

$$u(t, x + 2\pi) = u(t, x), \tag{2}$$

where $\varepsilon$ is a small positive parameter.

**Theorem 1.** If $\omega_0 > 0$, $\alpha > 0$, $\gamma > 0$, $d_0 < 0$ and the condition $\alpha > \gamma n^2$ is satisfied for some $n \in \mathbb{Z}$. Then for some $\varepsilon_0 > 0$, $0 < \varepsilon < \varepsilon_0$, the periodic on $t$ solutions

$$u_n = u_n(t, x) = \sqrt{\varepsilon} r_n \exp(i(\chi_n(\varepsilon)t + nx)) + O(\varepsilon)$$
of the boundary value problem (1), (2) exists. Here

\[ r_n = \sqrt{(\alpha - n^2 \gamma)} |d_0|^{-1}, \]

\[ \chi_n(\varepsilon) = \omega_0 + \varepsilon \beta + \varepsilon c_0 r_n^2 - \varepsilon \delta n^2, \]

\( n \in \mathbb{Z}. \)

These solutions are exponentially orbitally stable if and only if the condition

\[ (d_0 r_n^2 - \gamma k^2)^2 (\gamma^2 k^2 - \delta^2 n^2 - 2 \gamma d_0 r_n^2 - 4 \gamma^2 n^2 - 2 \delta c_0 r_n^2) > 4 \gamma^2 n^2 (c_0 r_n^2 - \delta k^2)^2 \]

is satisfied for all \( k \in \mathbb{Z} \setminus \{0\}. \)

Consider the equation of spin combustion

\[
\frac{\partial^2 \xi}{\partial t^2} + \xi = 2\varepsilon \left[ \frac{\partial \xi}{\partial t} \left( 1 - \frac{4}{3} \left( \frac{\partial \xi}{\partial t} \right)^2 \right) + \frac{1}{\varrho^2} \frac{\partial^3 \xi}{\partial t \partial x^2} \right], \quad \xi(t, x+2\pi) = \xi(t, x),
\]

(3)

where \( \varepsilon \) is a small positive parameter, \( \varrho > 0 \). There exist the traveling waves

\[ \xi_n(t, x) = \sqrt{1 - \frac{n^2}{\varrho^2}} \cos(t + nx) + O(\varepsilon) \]

of the problem (3), where \( n \in \mathbb{Z}, n^2 < \varrho^2 \). The traveling waves \( \xi_n(t, x) \) are exponentially orbitally stable if and only if the conditions \( n^2 < \frac{1}{16}(2\varrho^2 + 1) \) are fulfilled.

Consider the Brusselator equations with weak diffusion

\[ \frac{\partial u_1}{\partial t} = A - (B + 1) u_1 + u_1^2 u_2 + \varepsilon d_1 \frac{\partial^2 u_1}{\partial x^2}, \]

\[ \frac{\partial u_1}{\partial t} = B u_1 - u_1^2 u_2 + \varepsilon d_2 \frac{\partial^2 u_2}{\partial x^2} \]

with periodic boundary conditions

\[ u_1(t, x+2\pi) = u_1(t, x), \quad u_2(t, x+2\pi) = u_2(t, x), \]

where \( B = 1 + A^2 + \varepsilon, d_1 > 0, d_2 > 0, \varepsilon \) is a small positive parameter. If \( A > 0 \) and the condition \( 1 > (d_1 + d_2) n^2 \) is satisfied for some \( n \in \mathbb{Z} \), then for some \( \varepsilon_0 > 0, 0 < \varepsilon < \varepsilon_0 \), the periodic on \( t \) solutions
$u_1 + iu_2 = \sqrt{\varepsilon} r_n \exp(i(At + nx)) + O(\varepsilon)$ of the Brusselator model exists. Here $r_n^2 = \frac{4A^2}{A^2 + 2} (1 - (d_1 + d_2)n^2)$, $n \in \mathbb{Z}$.

Let $\mathbb{R}^n$ be $n$-dimensional space with the norm $|u| = \sqrt{u_1^2 + ... + u_n^2}$, $\mathbb{C} = \mathbb{C}[-\Delta, 0]$ be the space of functions, continuous on $[-\Delta, 0]$ with values in $\mathbb{R}^n$ with the norm $||\varphi|| = \sup_{-\Delta \leq \vartheta \leq 0} |\varphi(\vartheta)|$. We denote by $u_t$ the element of space $\mathbb{C}$ defined by function $u_t(\vartheta, x) = u(t + \vartheta, x)$, $-\Delta \leq \vartheta \leq 0$.

We consider a parabolic system with delay and weak diffusion

$$\frac{\partial u}{\partial t} = \varepsilon D \frac{\partial^2 u}{\partial x^2} + L(\varepsilon) u_t + f(u_t, \varepsilon) \quad (4)$$

with periodic condition (2). Here $\varepsilon$ is a small positive parameter, $u \in \mathbb{R}^n$, $L(\varepsilon) : \mathbb{C} \to \mathbb{R}^n$ is a continuous linear operator, $f : \mathbb{C} \times [0, \varepsilon_0) \to \mathbb{R}^n$, $f(\varphi, \varepsilon) = O(||\varphi||^2)$ as $||\varphi|| \to 0$, the operator $f(\varphi, \varepsilon)$ is continuous in $\varepsilon$ and four times continuously differentiable in $\varphi$. Let us assume that the zero solution of (4) for $\varepsilon = 0$ is asymptotically stable. The existence and stability of an arbitrarily large finite number of cycles for the equation (4) were considered in [2].

References


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R-spaces and uniform limits of sequences of functions

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A metric space $(Y, d)$ is said to be an $R$-space, if for every $\varepsilon > 0$ there exists a continuous map $r_\varepsilon : Y \times Y \to Y$ with the following
properties:
\[ d(y, z) \leq \varepsilon \implies r_{\varepsilon}(y, z) = y, \]
\[ d(r_{\varepsilon}(y, z), z) \leq \varepsilon \]
for all \( y, z \in Y \).

Every convex subset \( Y \) of a normed space \((Z, \| \cdot \|)\) equipped with the metric induced from \((Z, \| \cdot \|)\) is an R-space, where the map \( r_{\varepsilon} \) is defined as
\[
    r_{\varepsilon}(y, z) = \begin{cases} 
        z + (\varepsilon/\|y - z\|) \cdot (y - z), & \|y - z\| > \varepsilon, \\
        y, & \text{otherwise}. 
    \end{cases}
\]

Let \( X \) and \( Y \) be topological spaces. A map \( f : X \to Y \) is a Baire-one map, if there is a sequence \((f_n)_{n=1}^\infty\) of continuous maps between \( X \) and \( Y \) such that \( \lim_{n \to \infty} f_n(x) = f(x) \) for every \( x \in X \). The collection of all Baire-one maps between \( X \) and \( Y \) is denoted by \( B_1(X, Y) \).

It is well-known [1] that the uniform limit of a sequence of Baire-one maps \( f_n : X \to \mathbb{R} \) belongs to the first Baire class. Since the uniform limit of a sequence of continuous maps \( f_n : X \to Y \) is a continuous map for any metric space \( Y \), it is natural to ask whether the similar fact is true for Baire-one maps. It was proved in [2] that there exist a subset of \( Y \subseteq \mathbb{R}^2 \) and a sequence of Baire-one functions \( f_n : [0, 1] \to Y \) which converges uniformly to a function \( f \notin B_1([0, 1], Y) \). However, R-spaces are favorable range spaces for the problem on uniform limit of Baire-one functions.

**Theorem 1.** Let \( X \) be a topological space, \((Y, d)\) be a metric R-space. Then the class \( B_1(X, Y) \) is closed under uniform limits.

A result in the same direction was proved in [2].

**Theorem 2.** Let \( X \) be a topological space, \((Y, d)\) be a metric path-connected and locally path-connected space. Then the class \( B_1(X, Y) \) is closed under uniform limits.

Therefore, it is actual to find relations between R-spaces and path-connected locally path-connected spaces.

**Theorem 3.** Every path-connected R-space is locally path-connected.

**Theorem 4.** The unit circle \( S^1 = \{z \in \mathbb{C} : |z| = 1\} \) is an R-space.

**Question 1.** Is Hawaiian Earring an R-space?
We introduce the notion of a category enriched in a monoidal category [1] via the example of metric spaces. Generalized metric spaces of Lawvere [2] are discussed in detail. For them Cauchy completeness (in the sense of Lawvere [2]) is equivalent to the property that every fundamental sequence converges to at least one point. For ordinary categories (enriched in Set) Cauchy completeness is equivalent to the property that every idempotent splits. Denote by Ab the monoidal category of abelian groups. An Ab-category is Cauchy complete iff it admits finite direct sums and every idempotent splits. Thus, the notion of completeness relates analysis and algebra.

References


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Homotopy properties of spaces of smooth functions on surfaces
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Let $M$ be a smooth ($C^\infty$) connected compact surface without boundary. Consider the action of the group of $C^\infty$ diffeomorphisms $D(M)$ on the space $C^\infty(M)$ of smooth functions on $M$ by the rule: the result of the right action of $h \in D(M)$ on $f \in C^\infty(M)$ is the composition $f \circ h$. Then for each $f \in C^\infty(M)$ one can define its stabilizer

$$S(f) = \{ h \in D(M) \mid f \circ h = f \},$$

and the orbit

$$O(f) = \{ f \circ h \mid h \in D(M) \}.$$

Endow $D(M)$ and $C^\infty(M)$ with $C^\infty$ Whitney topologies. Then these topologies yield certain topologies on the corresponding stabilizes and orbits of smooth functions $f \in C^\infty(M)$. Denote by $S_{id}(f)$ the identity path component of $S(f)$, and by $O_f(f)$ the path component of $O(f)$ containing $f$.

The aim these lectures is to describe the homotopy types of the connected components of $S(f)$ and $O(f)$ for the case when $f$ is a Morse function.

Let $f : M \to \mathbb{R}$ be a $C^\infty$ Morse function. Then the following statements hold.

1. Suppose $f$ has at least one saddle critical point. Then
   - $S_{id}(f)$ is contractible;
   - there exists a free action of a finite group $G$ on a $k$-torus $T^k = S^1 \times \cdots \times S^1$ such that $O_f(f)$ is homotopy equivalent to the quotient $T^k/G$.

2. Otherwise, when $f$ has no saddles,
   - $S_{id}(f)$ is homotopy equivalent to the circle;
there exists a free action of a finite group $G$ on some $T^k$ such that $O_f(f)$ is homotopy equivalent to $SO(3) \times T^k / G$.

We will also describe the geometrical meaning of the group $G$, the number $k$ and precise algebraic structure of the fundamental group $\pi_1 O_f(f)$ for the case when $M$ is orientable and distinct from $S^2$.

In fact, the obtained results also hold for a more general class maps $f$ from compact surfaces (possibly with boundary) to $\mathbb{R}$ or to the circle $S^1$.

References


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ON VARIANTS OF THE EXTENDED BICYCLIC SEMIGROUP
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We shall follow the terminology of [3, 10]. In our report all spaces are assumed to be Hausdorff.

A (semi)topological semigroup is a topological space with a (separately) continuous semigroup operation. A topology $\tau$ on a semigroup $S$ is called:

- **shift-continuous** if $(S, \tau)$ is a semitopological semigroup;
- **semigroup** if $(S, \tau)$ is a topological semigroup.

The bicyclic semigroup (or the bicyclic monoid) $C(p,q)$ is the semigroup with the identity 1 generated by two elements $p$ and $q$ subject only to the condition $pq = 1$. The bicyclic monoid $C(p,q)$ is a combinatorial bisimple $F$-inverse semigroup and it plays an important role in the algebraic theory of semigroups and in the theory of topological semigroups.

The bicyclic semigroup admits only the discrete semigroup topology and if a topological semigroup $S$ contains it as a dense subsemigroup then $C(p,q)$ is an open subset of $S$ [4]. Bertman and West in [1] extend this result for the case of Hausdorff semitopological semigroups. Amazing dichotomy for the bicyclic monoid with adjoined zero $C^0 = C(p,q) \sqcup \{0\}$ was proved in [7]: every Hausdorff locally compact semitopological bicyclic semigroup with adjoined zero $C^0$ is either compact or discrete.

An interassociate of a semigroup $(S, \cdot)$ is a semigroup $(S, \ast)$ such that for all $a, b, c \in S$, $a \cdot (b \ast c) = (a \cdot b) \ast c$ and $a \ast (b \cdot c) = (a \ast b) \cdot c$. This definition of interassociativity was studied extensively in 1996 by Boyd, Gould, and Nelson in [2]. Certain classes of semigroups are known to give rise to interassociates with various properties. For example, it is very easy to show that if $S$ is a monoid, every interassociate must satisfy the condition $a \ast b = a \cdot c \cdot b$ for some fixed element $c \in S$ (see [2]). This type of interassociate was called a **variant** by
Hickey [9]. A general theory of variants has been developed by a number of authors. In the paper [6] the bicyclic semigroup \( \mathcal{C}(p, q) \) and its interassociates \((\mathcal{C}(p, q), *_{m,n})\) are investigated.

By \( \mathbb{Z} \) we denote the sets of all integers. On the Cartesian product \( \mathcal{C}_\mathbb{Z} = \mathbb{Z} \times \mathbb{Z} \) we define the semigroup operation as follows:

\[
(a,b) \cdot (c,d) = \begin{cases} 
 (a - b + c, d), & \text{if } b \leq c; \\
 (a, d + b - c), & \text{if } b > c,
\end{cases}
\]

for \( a, b, c, d \in \mathbb{Z} \). The set \( \mathcal{C}_\mathbb{Z} \) with such defined operation is called the extended bicyclic semigroup [11]. In the paper [5] algebraic properties of \( \mathcal{C}_\mathbb{Z} \) were described and it was proved therein that every non-trivial congruence \( \mathcal{C} \) on the semigroup \( \mathcal{C}_\mathbb{Z} \) is a group congruence, and moreover the quotient semigroup \( \mathcal{C}_\mathbb{Z} / \mathcal{C} \) is isomorphic to a cyclic group. Also it was shown that the semigroup \( \mathcal{C}_\mathbb{Z} \) as a Hausdorff semitopological semigroup admits only the discrete topology and the closure \( \text{cl}_T (\mathcal{C}_\mathbb{Z}) \) of the semigroup \( \mathcal{C}_\mathbb{Z} \) in a topological semigroup \( T \) was studied.

In the paper [8] we studied semitopological interassociates

\((\mathcal{C}(p, q), *_{m,n})\)

of the bicyclic monoid \( \mathcal{C}(p, q) \) for arbitrary non-negative integers \( m \) and \( n \). In particular, we showed that for arbitrary non-negative integers \( m, n \) and every Hausdorff topology \( \tau \) on \( \mathcal{C}_{m,n} \) such that \((\mathcal{C}_{m,n}, \tau)\) is a semitopological semigroup, is discrete. Also, we proved that if an interassociate of the bicyclic monoid \( \mathcal{C}_{m,n} \) is a dense subsemigroup of a Hausdorff semitopological semigroup \((S, \cdot)\) and \( I = S \setminus \mathcal{C}_{m,n} \neq \emptyset \) then \( I \) is a two-sided ideal of the semigroup \( S \) and show that for arbitrary non-negative integers \( m, n \), any Hausdorff locally compact semitopological semigroup \( \mathcal{C}^0_{m,n} (\mathcal{C}^0_{m,n} = \mathcal{C}_{m,n} \cup \{0\}) \) is either discrete or compact.

We describe the group \( \text{Aut} (\mathcal{C}_\mathbb{Z}) \) of automorphisms of the extended bicyclic semigroup \( \mathcal{C}_\mathbb{Z} \) and study a variant \( \mathcal{C}^m_{\mathbb{Z}, n} = (\mathcal{C}_\mathbb{Z}, *_{m,n}) \) of the extended bicycle semigroup \( \mathcal{C}_\mathbb{Z} \), where \( m, n \in \mathbb{Z} \), which is defined by the formula

\[
(a, b) *_{m,n} (c, d) = (a, b) \cdot (m, n) \cdot (c, d).
\]

We prove that \( \text{Aut} (\mathcal{C}_\mathbb{Z}) \) is isomorphic to the additive group of integers, the extended bicyclic semigroup \( \mathcal{C}_\mathbb{Z} \) and every its variant are
not finitely generated, describe the subset of idempotents $E(C^m,n)$ and Green’s relations on the semigroup $C^m,n$, and show that any two variants of the extended bicyclic semigroup $C_Z$ are isomorphic. At the end we discuss shift-continuous Hausdorff topologies on the variant $C_Z^{0,0}$. In particular, we prove that if $\tau$ is a Hausdorff shift-continuous topology on $C_Z^{0,0}$ then each of inequalities $a > 0$ or $b > 0$ implies that $(a, b)$ is an isolated point of $(C_Z^{0,0}, \tau)$ and construct an example a Hausdorff semigroup topology $\tau^*$ on the semigroup $C_Z^{0,0}$ such that all its points with the properties $ab \leq 0$ and $a + b \leq 0$ are not isolated in $(C_Z^{0,0}, \tau^*)$.

References


Superfractality of an incomplete sums set of a certain positive series

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The achievement set (or a partial sumset) of the series
\[ \sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots = S_n + (a_{n+1} + a_{n+2} + \ldots) = S_n + r_n \]
is called a set
\[ E\{a_n\} \equiv \left\{ x : x = x(M) = \sum_{n \in M \subset \mathbb{N}} a_n, \ M \in 2^{\mathbb{N}} \right\}, \]
and every its element — an incomplete sum of the series.

We consider a family of convergent positive normed series with real terms defined by conditions
\[ \sum_{n=1}^{\infty} d_n = c_1 + \ldots + c_1 + c_2 + \ldots + c_2 + \ldots + c_n + \ldots + c_n + \tilde{r}_n = 1, \]
where \((a_n)\) is a nondecreasing sequence of real numbers. For a partial case \((a_n) = 2^{n-1}, c_n = (n+1)\tilde{r}_n, \ n \in \mathbb{N}\), the geometry of a series (i.e. properties of cylindrical sets, metric relations generated by them, and topological and metric properties of the set of all incomplete sums of a series) are investigated.

We consider a random variable
\[ \xi = \sum_{n=1}^{\infty} d_n \xi_n, \]where \((\xi_n)\) — sequence of independent random variables with distribution:
\[ P\{\xi_n = 0\} = p_{0n} \geq 0, \ P\{\xi_n = 1\} = p_{1n} \geq 0, \ p_{0n} + p_{1n} = 1. \]
Theorem 1. The distribution of the random variable $2$, defined by the series $1$, is pure. It is purely discrete if and only if

$$
M = \prod_{k=1}^{\infty} \max\{p_{0k}, p_{1k}\} > 0.
$$

(4)

In the case of discreteness of the distribution of the random variable $2$ its point spectrum consists of a point

$$
x_0 = \sum_{n=1}^{\infty} \alpha_n^* d_n, \quad p_{\alpha_n^* n} \geq p[1-\alpha_n^* n],
$$

and all points $x$ which can be represented in the form:

$$
x = \sum_{n=1}^{m} \alpha_n d_n + \sum_{n=m+1}^{\infty} \alpha_n^* d_n,
$$

where $\alpha_n \in \{0, 1\}$, $p_{\alpha_n n} \neq 0$ for $n \leq m$.

By the spectrum $S_\xi$ of the distribution of a random variable $\xi$ we mean the set of growth points of its distribution function $F_\xi(x)$, i.e.

$$
S_\xi = \{x : F_\xi(x + \varepsilon) - F_\xi(x - \varepsilon) = P\{\xi \in (x - \varepsilon; x + \varepsilon)\} > 0, \forall \varepsilon > 0\}.
$$

Lemma 2. If $p_{in} > 0$ for all $i \in \{0, 1\}$ and all $n \in \mathbb{N}$, then the spectrum $S_\xi$ of the distribution of the random variable $\xi$ coincides with the set $E\{d_n\}$ of all subsums (incomplete sums) of the series (1), that is,

$$
S_\xi = E\{d_n\} \equiv \left\{x : x = \sum_{n \in M} d_n, \quad M \in 2^{\mathbb{N}}\right\}.
$$

Corollary 3. For the spectrum $S_\xi$ of the distribution of a random variable $\xi$ the following inclusion holds: $S_\xi \subset E\{d_n\}$.

Lebesgue measure zero subsets of $\mathbb{R}$ with the Haussdorff-Besikovich dimension 1, are called superfractal.

Theorem 4. The set of incomplete sums (1) is a superfractal set.
Corollary 5. The spectrum $S_\xi$ of the distribution of a random variable $\xi$ is a superfractal set.

Theorem 6. In the case of continuity ($M = 0$), the distribution of the random variable $\xi$ is a singular Cantor type distribution with a superfractal spectrum.

An autoconvolution of the distribution of a random variable $\xi$ is the distribution of the random variable $\psi_2 = \xi^{(1)} + \xi^{(2)}$, and an $s$-multiple convolution of the distribution of a random variable $\xi$ is the distribution of a random variable

$$\psi_s = \xi^{(1)} + \xi^{(2)} + ... + \xi^{(s)},$$

where $\xi^{(j)}$ are independent and equally distributed random variables each having the same distribution as $\xi$.

Lemma 7. The spectrum $S_{\psi_s}$ of the distribution of a random variable $\psi_s$ is a subset of the interval $[0, s]$ and belongs to the union

$$\bigcap_{n=0}^{\infty} (s \cdot 2^k + 1)$$

of isometric segments of length $s \tilde{r}_n$, $n = 1, 2, 3, \ldots$.

Theorem 8. In the case of the continuity of the random variable $\xi$ ($M = 0$) the distribution of the random variable $\psi_s$, for any natural $s \geq 2$ is a singular Cantor type distribution with a superfractal spectrum.

References


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On invariant max-plus closed convex sets of idempotent measures

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In [4], the authors defined the notion of an invariant idempotent measure under given IFS on a complete metric space. These measures are idempotent counterparts of the probabilistic fractals [1].

In [2], [3], the first-named author considered invariant objects under given IFS on a complete ultrametric space, and obtained the construction of an invariant max-plus closed convex set of idempotent measures.

Recall that an idempotent measure on a compact Hausdorff space $X$ is a functional $\mu : C(X) \to \mathbb{R}$ that preserves constants, the maximum operation and is weakly additive (i.e., preserves sums in which at least one summand is a constant function) [5]. Given an arbitrary metric space $X$, we denote by $I(X)$ the set of idempotent measures of compact support on $X$. A nonempty subset $A \subset I(X)$ is called max-plus convex if $\max\{t + \mu, \nu\} \in A$ for every $\mu, \nu \in A$ and $t \in [-\infty, 0]$.

The aim of the talk is to obtain a counterpart for the construction of an invariant max-plus closed convex set of idempotent measures under given IFS on a complete metric space. For a complete metric space $X$, we consider the space $ccI(X)$ (of closed convex subsets of idempotent measures of compact support on $X$) as a subspace of $C'(X)$ and use the weak* convergence to prove the existence of an invariant element.

References


Development of Hahn’s theorem on intermediate function

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We call a pair of functions \((g, h)\) a Hahn’s pair on topological space \(X\), if \(g : X \to \mathbb{R}\) and \(h : X \to \mathbb{R}\) are upper and lower semicontinuous functions such that \(g(x) \leq h(x)\) on \(X\). If \(g(x) < h(x)\) on \(X\), then \((g, h)\) is a strict Hahn’s pair. Function \(f : X \to \mathbb{R}\) is called intermediate for Hahn’s pair \((g, h)\) on \(X\), if \(g(x) \leq f(x) \leq h(x)\) on \(X\) and strictly intermediate, if \(g(x) < f(x) < h(x)\) whenever \(g(x) < h(x)\) and \(g(x) = f(x) = h(x)\) when \(g(x) = h(x)\). According to Hahn-Dieudonne-Katetov-Tong theorem [1, p. 105] \(T_1\)-space \(X\) is normal iff each Hahn’s pair \((g, h)\) on \(X\) has an intermediate continuous function. This theorem has a lot of analogues (see [2] and the literature given there). Here we present some of our recent results regarding differentiable intermediate function and intermediate function for Hahn’s pair of separately semicontinuous functions.

Theorem 9. Let \(X\) be a separable Hilbert space and \((g, h)\) be a Hahn’s strict pair on \(X\). Then there is a \(C^\infty\)-function \(f : X \to \mathbb{R}\) which is strictly intermediate for \((g, h)\).

This result is also true for Asplund spaces and parallelepipeds in \(\mathbb{R}^n\).

For a map \(f : X \times Y \to Z\) and a point \((x, y) \in X \times Y\) we write \(f^x(y) = f(x, y) = f_y(x)\). For topological spaces \(X, Y\) and \(Z\) we denote by \(C(X), C^u(X)\) and \(C^t(X)\) the spaces of continuous and
upper or lower respectively semicontinuous functions \( f : X \to \mathbb{R} \), by 
\( CC(X \times Y), C^uC^u(X \times Y) \) and \( C^lC^l(X \times Y) \) — spaces of separately 
continuous and upper or lower separately semicontinuous functions 
\( f : X \times Y \to \mathbb{R} \), and by \( C(X,Y) \) and \( CC(X \times Y, Z) \) — spaces of 
continuous maps \( f : X \to Y \) and separately continuous maps \( f : X \times Y \to Z \), respectively.

For topological spaces \( X \) and \( Y \) the ordered pair \((g, h)\) of functions 
\( g \in C^uC^u(X \times Y) \) and \( h \in C^lC^l(X \times Y) \) is called a separate Hahn’s pair.

Let us recall, that plus-topology \( C \) on product \( X \times Y \) of two topo-
logical spaces consists of sets \( O \subseteq X \times Y \) such that for each point 
\( p = (x, y) \in O \) there exist neighborhoods \( U \) of point \( x \) and \( V \) of point 
\( y \) in spaces \( X \) and \( Y \), respectively, with \((U \times \{y\}) \cup (\{x\} \times V) \subseteq O\).

**Theorem 10.** Let \( X \) and \( Y \) be \( T_1 \)-spaces. Then each separate Hahn’s pair \((g, h)\) on product \( X \times Y \) has an intermediate separately continuous function if and only if space \( Q = (X \times Y, C) \) is normal.

**References**


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**On discontinuity points set of separately continuous functions**

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For topological spaces \( X \) and \( Y \) and a mapping \( f : X \to Y \) by 
\( D(f) \) we denote the discontinuity points set of \( f \).
We say that a subset $A$ of the product $X \times Y$ of topological spaces $X$ and $Y$ is \textit{locally projectively nowhere dense} (meager), if for every $x \in X$ there exists a neighborhood $U$ of $x$ in $X$ such that the set $A \cap U$ has nowhere dense (meager) projections on $X$ and $Y$.

Function $f : X \times Y \to \mathbb{R}$ is called \textit{separately continuous}, if it continuous with respect to each variable.

**Theorem 1.** Let $X$ and $Y$ be metrizable spaces and $E \subseteq X \times Y$. Then there exists a separately continuous function $f : X \times Y \to \mathbb{R}$ with $D(f) = E$ if and only if the set $E$ is the union of a sequence of a $F_{\sigma}$-sets which are locally projectively meager.

**Theorem 2.** Let $X = \prod_{s \in S} X_s$ and $Y = \prod_{t \in T} Y_t$ be topological products of families of metrizable separable spaces $X_s$ and $Y_t$ respectively. Then a set $E \subseteq X \times Y$ is the discontinuity points set for a separately continuous function $f : X \times Y \to \mathbb{R}$ if and only if there exist at most countable sets $S_0 \subseteq S$ and $T_0 \subseteq T$ and projectively meager $F_{\sigma}$-set $E_0$ in a product $X_0 \times Y_0$, where $X_0 = \prod_{s \in S_0} X_s$ and $Y_0 = \prod_{t \in T_0} Y_t$, such that $E = pr^{-1}(E_0)$ where $pr : X \times Y \to X_0 \times Y_0$ is the natural projection for which $pr(x, y) = (x|_{S_0}, y|_{T_0})$.

**Theorem 3.** Let $X, Y$ be \v{C}ech complete spaces, $(E_n)_{n=1}^{\infty}$ be a sequence of separable compact perfect projectively nowhere dense $G_\delta$-sets $E_n$ in $X \times Y$ and $E = \bigcup_{n=1}^{\infty} E_n$. Then there exists a separately continuous function $f : X \times Y \to \mathbb{R}$ such that $D(f) = E$.

**Theorem 4.** Let $X, Y$ be \v{C}ech complete spaces, $(A_n)_{n=1}^{\infty}$, $(B_n)_{n=1}^{\infty}$ be sequences of nowhere dense compact $G_\delta$-sets $A_n$ and $B_n$ in $X$ and $Y$ respectively, $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} B_n$. Then there exists a separately continuous function $f : X \times Y \to \mathbb{R}$ such that $pr_X D(f) = A$ and $pr_Y D(f) = B$.

**Example 1.** There exist Eberlein compacts $X$ and $Y$ and nowhere dense zero sets $A$ and $B$ in $X$ and $Y$ respectively such that $D(f) \neq A \times B$ for every separately continuous function $f : X \times Y \to \mathbb{R}$.

**Example 2 (CH).** There exist separable Valdivia compacts $X$ and $Y$, nowhere dense separable zero sets $E$ and $F$ in $X$ and $Y$ respectively such that $D(f) \neq E \times F$ for every separately continuous function $f : X \times Y \to \mathbb{R}$.
**Question 1.** To characterize of discontinuity points sets of separately continuous functions on the product of two Eberlein compacts.

**Question 2.** Let $X$ be a metrizable compact, $Y$ be an Eberlein compact and $E$ be a projectively nowhere dense zero set in the product $X \times Y$. Does there exist a separately continuous function $f : X \times Y \rightarrow \mathbb{R}$ with $D(f)=E$?

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**Dimensions of hyperspaces and non-additive measures**

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**Topological invariants of BPS states and Higgs coupling measurements at the LHC**

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Since only the Higgs boson was discovered among the theoretically predicted particles, the further development of physics is associated with the study of the properties of this particle. Standard Model (SM) has a few pressing questions:

- the baryon asymmetry of the Universe;
• the large mass hierarchy;
• the prevalence of the Dark Matter in the Universe;
which call for physics beyond the SM.

The most promising methods for studying of new physics are the experiments connected with the properties of the Higgs boson coupling [1, 2]. As these experiments assume the formation of new particles beyond SM, there is a need for a new theory. Mathematical calculations of possible observables in the framework of such theories would be a serious help for modern physics.

One of such methods is to consider vector bundles with a four-dimensional base and a fiber – vector space with the structure group SU(5). As the problem of Higgs selfcoupling is connected with the vacuum stability, it would be appropriate to consider the instanton numbers associated with tunnel transitions between vacuum states and with different topological quantum numbers characterizing the BPS states. The BPS solitons in SUSY theories with codimension four, three, two, and one are called instantons, monopoles, vortices, and domain walls, respectively [3]. The theory has a stable domain wall interpolating between two different vacua and the energy density of the wall is nothing but the value of the central charge $q$, related to the integral

$$J^{\mu\nu} = \int d^3x \varepsilon^{0\mu\nu\rho} \partial_\rho W(S^2) \rightarrow [W(S^2)_{z\rightarrow +\infty} - W(S^2)_{z\rightarrow -\infty}] ,$$

with superpotential

$$W = \frac{\Lambda^5}{S^2} + \frac{m}{4} S^2 ,$$

where $\Lambda$ is the scale parameter of the SU(2) gauge theory, $m = \sqrt{2\lambda v}$, $S$ is composite operator of SU(2) group, [4]. As superpotential has a crucial role for the properties of SUSY particles [5], it is necessary to draw a conclusion about the importance of the BPS characteristics connected with topological invariants such as central charge and new particle spectra beyond the SM.
References


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ON NARROW OPERATORS

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The lectures will consist of two parts: 1) An introduction to narrow operators; 2) Some open problems on narrow operators.

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THE SUM OF CONSECUTIVE FIBONACCI NUMBERS

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The sums of the sequential Fibonacci numbers and some other sequences form new sequences. We consider such task: find all the natural values $m > 1$ such that the divisibility is fulfilled $\sum_{k=1}^{m} f_k : m$, where
sequence \( \{f_n\}_{n=1}^{\infty} \) is determined by the ratio \( f_{n+1} = f_n + f_{n-1}, \ n \geq 2 \), values \( f_1 \) and \( f_2 \) - some integers that may vary. Let’s consider examples of calculations.

1) Let \( f_1 = 1 \) and \( f_2 = 1 \), so we consider the Fibonacci sequence. In this case we get such values:

\[ 1, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, \ldots \]

2) Let \( f_1 = 2 \) and \( f_2 = 1 \). In that case we get such values \( m \):

\[ 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, \ldots \]

Here was used the equality: \( \sum_{k=1}^{n+m} f_k = f_{n+m+2} - 1 \). The following statement takes place.

**Theorem 11.** The sum of any \( m \) sequential Fibonacci numbers is multiple of \( m \) and only if the numbers \( f_m \) and \( f_{m+1} - 1 \) are multiples of \( m \).

**Lemma 12.** Let \( f_{n+1}, f_{n+2}, \ldots, f_{n+m}, n \geq 0 - m \) arbitrary successive Fibonacci numbers. Then the next equality holds:

\[
\sum_{k=1}^{m} f_{n+k} = f_{n+m+2} - f_{n+2}
\]

Indeed, taking into account the equality \( \sum_{k=1}^{l} f_k = f_{l+2} - 1 \), for all natural numbers \( l \), we will make the following transformations:

\[
\sum_{k=1}^{m} f_{n+k} = f_{n+1} + f_{n+2} + \ldots + f_{n+m} = \\
= (f_1 + f_2 + \ldots + f_{n+m}) - (f_1 + f_2 + \ldots + f_n) =
\]

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\[
= \sum_{k=1}^{n+m} f_k - \sum_{k=1}^{n} f_k = (f_{n+m+2} - 1) - (f_{n+2} - 1) = f_{n+m+2} - f_{n+2}.
\]

The criterion of divisibility of the sum of successive Fibonacci numbers by the number terms has the following form:

**Theorem 13.** The sum of any \( m \) sequential Fibonacci numbers is multiple of \( m \) if and only if the numbers \( f_m \) and \( f_{m+1} - 1 \) are multiples of \( m \).

We denote by \( T(m) \) the period of mentioned above sums of consecutive Fibonacci numbers reduced by the modulo \( m \).

**Statement 1.** If \( \left( \frac{5}{p} \right) = 1 \) then the period \( T(p) \) divides \( p-1 \) if \( \left( \frac{5}{p} \right) = -1 \) then \( T(p) \) divides \( 2p + 2 \).

**Theorem 14.** If \( m = pq \), where \( p, q \in P \) then the period is equal to \( T(p)T(q) \) viz \( T(pq) = lcm(T(p), T(q)) \).

**Example 15.** The period by a composite module 33 is equal to 33 according to formula \( T(33) = LCM(8, 10) = 40 \), where \( 8 = T(3) \), \( 9 = T(11) \).

Other periods are the following: \( T(72) = T(9 \cdot 8) = T(9) \cdot T(8) = lcm(8, 18) = 24 \cdot 3 \). The period by the composite module \( m = 2 \cdot 31 \) is the product \( T(2) = 3 \) by \( T(31) = 30 \). In other words \( T(62) = T(2)T(31) = 30 \).

**Example 16.** The period by a composite module 38 is equal to according to \( T(38) = T(2 \cdot 19) = lcm(3, 18) = 18 \)

**Theorem 17.** If \( m = p^2 \) then the period of the sums \( T(p) \) is equal to \( pT(p) \).

**Example 18.** Take into account that \( T(7) = 16 \), and as an example we check that \( T(7^2) = T(7)7 \). Indeed due calculations we obtain the following

\[
T(49) = 112 = 2^47 = T(7^2) = T(7)7 = (16) \cdot 7 = (2^4)7.
\]
The concept of numerical index was introduced by G. Lumer in 1968 in the context of the study and the classification of operator algebras.

This is a constant of a Banach space relating the behaviour of the numerical range with that of the usual norm on the Banach algebra of all bounded linear operators on the space. Recently, Ardalani introduced new concepts of numerical range and numerical radius of one operator with respect to another one, which generalize in a natural way the classical concepts of numerical range and numerical radius. The aim of this talk is to study basic properties of these new concepts, present some examples and provide results on the stability of the numerical index with respect to an operator under some natural operations such as absolute sums or $c_0$, $\ell_1$ and $\ell_\infty$-sums and on computing the numerical index with respect to an operator of some vector-valued function spaces.
The Bishop-Phelps-Bollobás property  
and some related topics

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The origin of the study of norm-attaining operators between Banach Spaces is the Bishop-Phelps Theorem [2] about the density of norm-attaining functionals. In 1961, E. Bishop and R. R. Phelps proved that given any Banach Space $X$ over the field $\mathbb{K}$ of real or complex numbers, the set of functionals over $X$ (this is, bounded linear operators from $X$ to $\mathbb{K}$) is dense in the topological dual of $X$, denoted by $X^*$. In 1963, J. Lindenstrauss [5] studied whether the result remained true when we consider norm-attaining operators between two Banach Spaces $X$ and $Y$, instead of functionals, and gave a negative answer to the question. As of that moment, many papers have been published worldwide studying this topic.

In 1970, B. Bollobás [3] gave a refined version of Bishop-Phelps Theorem, with which one can approximate at the same time a functional and a vector in which it almost attains its norm by a norm-attaining functional and a vector in which it attains its norm. This is the starting point of the Bishop-Phelps-Bollobás property, introduced in 2008 by M. D. Acosta, R. M. Aron, D. García and M. Maestre [1]. The Bishop-Phelps-Bollobás property (the BPBp for short) is just an adaptation of Bollobás result to operators between Banach Spaces instead of functionals. Clearly, it is a stronger property than the density
of norm-attaining operators, and actually, it is known that they are not equivalent. Ever since the property was introduced, there have been numerous papers written studying this topic and similar ones. In 2013, A. J. Guirao and O. Kozhushkina [4] adapted the BPBp for operators that attain their numerical radius instead of their norm, and that started another field of study in Functional Analysis. In this talk we will see some of the main results that have been found about these topics.

References


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An analog of the Schwarz lemma for regular homeomorphisms

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Let $B = \{ z \in \mathbb{C} : |z| < 1 \}$ and $f: \mathbb{B} \to \mathbb{C}$ be a regular homeomorphism of Sobolev class $W_{1,1}^{1,1}$. For any $p > 1$, the quantity

$$D_p(z) = D_p(re^{i\theta}) = \frac{|f_\theta(re^{i\theta})|^p}{r^p J_f(re^{i\theta})}$$

is called $p$-angular dilatation of $f$ at a point $z \in B$, $z \neq 0$, with respect to the origin.

Let $B_r = \{ z \in \mathbb{C} : |z| < r \}$, $\gamma_r = \{ z \in \mathbb{C} : |z| = r \}$. For $p > 1$, denote

$$d_p(r) = \left( \frac{1}{2\pi r} \int_{\gamma_r} D_{p-1}^1(z)|dz| \right)^{p-1}.$$

**Theorem 1.** Let $f: \mathbb{B} \to \mathbb{B}$ be a regular homeomorphism of Sobolev class $W_{1,1}^{1,1}$ possessing Lusin’s $(N)$-property, and $f(0) = 0$. Suppose that there exists $k$ such that for $p > 2$,

$$\liminf_{r \to 0} \left( \frac{1}{\pi r^2} \iint_{B_r} D_{p-1}^1(z) dxdy \right)^{p-1} \leq k < \infty.$$

Then

$$\liminf_{z \to 0} \frac{|f(z)|}{|z|} \leq c_p k^{\frac{1}{p-2}} < \infty,$$

where $c_p$ is a positive constant depending only on $p$.

**Theorem 2.** Let $f: \mathbb{B} \to \mathbb{B}$ be a regular homeomorphism of Sobolev class $W_{1,1}^{1,1}$ possessing Lusin’s $(N)$-property and normalized by $f(0) = 0$. Suppose that $p > 2$ and there exists $k_0$ such that

$$k_0 = \limsup_{r \to 0} r^{p-2} \int_r^1 \frac{dt}{t^{p-1}d_p(t)}.$$

Then

$$\liminf_{z \to 0} \frac{|f(z)|}{|z|} \leq (p - 2)^{\frac{1}{p-2}} k_0^{\frac{1}{p-2}}.$$
A GENTLE INTRODUCTION TO FORCING

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BOREL SETS IN TOPOLOGICAL SPACES

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We present a construction of the Borel hierarchy in general topological spaces and its relation to Baire hierarchy. We define mappings of Borel class $\alpha$, prove the validity of the Lebesgue–Hausdorff–Banach characterization for them and show their relation to Baire classes of mappings on compact spaces. We prove a key theorem on invariance of Borel sets with respect to perfect mappings. The obtained results are used for studying Baire and Borel order of compact spaces. We present several examples showing some natural limits of our results in non-compact spaces. We also include applications of the obtained results to the theory of compact convex sets.

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THE FINDING OF THE NUMBER OF THE NONISOMORFIC $(n,m)$-GRAPHS

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The simple graph $G$ with $n$ vertices and $m$ edges is called $(n,m)$-graph. Let $T(n,m)$ is the number of the nonisomorphic $(n,m)$-graph. In [3] we can find $T(n,m)$ for $n = 1,18$ and in [1] we can find the sequence A008406 for $n = 1,20$. In [2] the formula for the generating function of the number of the simple graphs has been obtained. This work deals with the simple way of the calculation of the some number $T(n,m)$ for $n = 21,26$.

Some properties:
1) Let $M$ is the set of all nonisomorphic $(n,m)$-graphs. Then the subset, which is equivalent to $M$, exists in the set of all nonisomorphic $(n+1,m)$-graphs. Arbitrary graph of the subset differs from any graph of the set by one isolated vertex.

2) If the graph with the vector of the degrees $(1,1,,1)$ exists in the set of all nonisomorphic $(n,m)$-graphs, then $n = 2m$.

3) Let $k$ is the number of all nonisomorphic $(2m,m)$-graphs. Then the number of all nonisomorphic $(n,m)$-graphs equals $k$ for all $n > 2m$.

**Definition 1.** Let $(s_1, s_2, ..., s_n)$ is the vector of the degrees of $(n,m)$-graph $G$. Let for any $i$ takes place $s_i \geq 2$ and $(v_i,v_j)$ is one of the edges. We add new isolated vertex $v_{n+1}$. We move off the edge $(v_i,v_j)$ and we add the edge $(v_j,v_{n+1})$. Then the degree of the vertex $v_i$ is reduced by 1 and the degree of the new vertex is 1. In this case we call the transformation of the graph $G$ $P$-transformation.

Using $P$-transformation we can transform $(2m-k,m)$-graph for $k > 0$ into $(2m,m)$-graph. The following formulas have been proved:

$$T(2m,m) = T(2m,1) + T(2m-1,m), m > 1$$

$$T(2m,m) = T(2m,2) + T(2m-2,m), m > 2$$

$$T(2m,m) = T(2m,3) + T(2m-3,m), m > 3$$

$$T(2m,m) = T(2m,4) + T(2m-4,m) + 1, m > 5$$

$$T(2m,m) = T(2m,5) + T(2m-5,m) + 4, m > 7$$

$$T(2m,m) = T(2m,6) + T(2m-6,m) + 4, m > 9$$

We calculate $T(21,11), T(22,11), T(21,12), T(22,12), T(23,12), T(24,12), T(21,13), T(22,13), T(23,13), T(24,13), T(25,13), T(26,13)$. 
References

[1] https://oeis.org


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GJMS operator on Einstein Riemannian manifold

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On topological aspects of the digital image segmentation

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Introduction. Nowadays the data mining needs new effective methods of data processing caused by the growth of data flow. The image segmentation is usufull part of a machine vision, an object detection, the recognition tasks and etc. The most of image segmentation methods are based on statistics methods that in fact lead to some disadvantages. For example, K-means algorithm requires to know the quantity of clusters in advance that restricts its application.

The authors propose the image segmentation algorithm based on the persistent homologies as the effective mode of topological data analysis.

Background. Terms of computational topology are formulated, see [1, 3, 4]. Given $S \subset Y$ and $\epsilon \in \mathbb{R}$, let $G_\epsilon = \{S, E_\epsilon\}$ be $\epsilon$-neighborhood graph on $S$, where $E_\epsilon = \{(u, v) : d(u, v) \leq \epsilon, u \neq v\}$
v ∈ S}. A clique in a graph is the subset of vertices that induces a complete subgraph. The clique complex has the maximal cliques of a graph as its maximal simplices.

The Vietoris-Rips complex $C_\epsilon$ is the clique complex of $\epsilon$-neighborhood graph.

A filtration of a space $X$ is a nested sequence of subspaces: $\emptyset \subseteq X_1 \subseteq \ldots \subseteq X_n = X$.

For $\epsilon \leq \epsilon'$, it is true that $C_\epsilon \subseteq C_{\epsilon'}$. The set $\{C_{\epsilon_i}\}_{i=1}^k$ of Vietoris-Rips complexes is the filtration for any finite set $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_k\}$, where $\epsilon_i < \epsilon_j$, $i < j$.

Let $H_p^i = H_p(C_{\epsilon_i})$, where $H_p$ be $p$–th homology, and $f_p^{i,j} : H_p^i \rightarrow H_p^j$, $i < j$, be a map.

The $p$–th persistent homology $H_{p}^{i,j}$ is $\text{Im} f_p^{i,j}$ for $0 \leq i < j \leq k + 1$. On other words, $H_{p}^{i,j} = Z_p^i / (B_p^j \cap Z_p^i)$, where $Z_p^i$ is $p$–cycles of $C_{\epsilon_i}$ and $B_p^j$ is $p$–boundaries of $C_{\epsilon_j}$. There is a method of their calculation based on the matrices algebra, the persistence barcode and the persistence diagrams (see [4]).

It’s known that $\beta_0 = \text{rank} H_{0}^{i,j}$ is the amount of connected components of the space. For the digital image segmentation, it is the same as the quantity of clusters.

Practice. Let consider a digital image $D$ as a set of points $M$ in $R^5$. Denote by $L$ the length of $D$ and $W$ the width of $D$. Every pixel $P$ has two parameters of the plane location ($x$ and $y$) and three color components (for example, RGB). The digital image segmentation algorithm based on a persistent homology is following:

1. Let $P(x, y, r, g, b)$ be a pixel of $D$. Then the coordinates of every $P$ are normalized and the following is obtained: $x' = \frac{x}{\max\{L, W\}}$, $y' = \frac{y}{\max\{L, W\}}$, $r' = \frac{r}{255}$, $g' = \frac{g}{255}$ and $b' = \frac{b}{255}$, where $x', y', r', g', b' \in [0; 1]$. The set $M$ is transformed into the new set $M'$ in $R^5$;

2. Fix a finite set $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_k\}$ such that $\epsilon_i < \epsilon_j$ for $i < j$. Construct the filtration of Vietoris-Rips complexes $\{C_{\epsilon_i}\}_{i=1}^k$ on the even grid of the set $M'$;

3. Construct the matrices of persistent homologies of $\{C_{\epsilon_i}\}_{i=1}^k$;
4. Calculate the rank of Smith normal form of persistent homologies matrices. This number is the quantity of segmentation clusters.

The image segmentation algorithm based on the persistent homologies is implemented in C# (.NET4.5) and the results are compared with the K-means algorithm.

**Conclusions.** After testing on real images it becomes obvious that the digital image segmentation algorithm based on the persistent homologies is more effective than K-means algorithm and not sensitive even to high noise levels.

**References**


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**NON-EXPANSIVE BIJECTIONS AND EC-PLASTICITY OF THE UNIT BALLS**

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A metric space $E$ is said to be expand-contract plastic (or briefly an EC-space) if every non-expansive bijection $F: E \to E$ is an isometry. The EC-plasticity of totally bounded metric spaces was established in [4, Theorem 1.1]. In particular, the unit ball or even every bounded set of any finite-dimensional Banach space is EC-plastic. However, the question about plasticity of the unit ball of an arbitrary Banach space is open. In [1, Theorem 2.6] the mentioned property is proved for the unit balls of strictly convex Banach spaces. So, the unit balls of the spaces $L_p$ with $1 < p < \infty$ as well as of all Hilbert spaces are plastic. On the other hand, there is an example of bounded closed convex set in an infinite-dimensional Hilbert space that is not EC-space [1, Example 2.7]. As we can see, the question about plasticity in infinite-dimensional spaces is much more difficult. We answered this question in positive for the unit ball of the space $\ell_1$, which is not strictly convex.

**Theorem 1** ([2, Theorem 1]). The unit ball of $\ell_1$, is an EC-space.

Another interesting question connected with the previous one is about unit balls of two different Banach spaces $X$ and $Y$. If one consider a non-expansive bijection $F: B_X \to B_Y$, for which spaces $X$ and $Y$ the mapping $F$ appears to be an isometry? In this field we received some generalization. For cases when the unit ball of the space $Y$ is known to be EC-plastic, we get that $X$ may be an arbitrary space. To be more exact, the following theorems hold.

**Theorem 2** ([5, Theorem 3.1]). Let $F: B_X \to B_Y$ be a bijective non-expansive map. If $Y$ is strictly convex, then $F$ is an isometry.

**Theorem 3** ([5, Theorem 3.5]). Let $F: B_X \to B_{\ell_1}$ be a bijective non-expansive map. Then $F$ is an isometry.

**Theorem 4** ([5, Theorem 3.8]). Let $Y$ be a finite-dimensional Banach space, $F: B_X \to B_Y$ be a bijective non-expansive map. Then $F$ is an isometry.

We have also done one more step in solving the problem with the unit balls of two different spaces.
Theorem 5 ([3, Theorem 3.1]). Let $X$ be a Banach space, $Z_i, i \in I$ be a fixed collection of strictly convex Banach spaces, $Z$ be the $\ell_1$-sum of the collection $Z_i, i \in I$, and $F : B_X \rightarrow B_Z$ be a non-expansive bijection. Then $F$ is an isometry.

As a consequence, we get the EC-plasticity of the unit ball of the space $Z$.

References


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Basics of proper forcing, with emphases on posets consisting of trees

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Absoluteness Theorems

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