Absoluteness theorems, part 3 (Szymon Żeberski)

Applications of Shoenfield theorem

Th. (Mazurkiewicz) Let $f_n : [0,1] \to [0,1]$, $n \in \omega$ be a sequence of Borel functions. Then there exists P a perfect set and $A \in [\omega]^\omega$ such that $\{f_n(x)\}_{n\in A}$ converges for $x \in P$.

p-f.

We enlarge our universe, by a generic extension, to one where $\mathfrak{s} > \omega_1$ (for instance MA)

($\mathfrak{s}$ – splitting number)

$\kappa < \mathfrak{s}$ is equivalent to:

- $\{0,1\}^\kappa$ is sequentially compact
- $(\{0,1\}^\omega)^\kappa$ is sequentially compact
- $[0,1]^\kappa$ is sequentially compact.

In our extension $\omega_1 < \mathfrak{s}$.

Take $X \subseteq [0,1]$ $|X| = \omega_1$

$(f_n \restriction X)_{n\in\omega}$ a sequence of elements in $[0,1]^X$

$[0,1]^X$ is sequentially compact. So

$\exists A \in [\omega]^\omega$ $(f_n \restriction X)_{n\in A}$ is convergent.

The set $\{x : (f_n(x))_{n\in A}$ is convergent$\} =$

$$= \{x : (\forall k)(\exists n)(\forall m,m')(m,m' > n \wedge m,m' \in A \to |f_m(x) - f_{m'}(x)| < \tfrac{1}{k})\}$$

is Borel and uncountable (contains X). Thus it contains a perfect subset.

We have proved that in extension:

$\exists P \in Perf([0,1])\ \exists A \in [\omega]^\omega\ \forall x \in [0,1]$

$$(x \in P \to (f_n(x))_{n\in A} \text{ is convergent})$$

The latter statement is Σ^1_2-sentence.
So, it is true in ground model. $\square$

Th. (Eggelston) Let $A \subseteq [0,1]^2$ be such that $\lambda^2(A) > 0$.
Then there exist $P, Q \subseteq [0,1]$ $P \in \text{Perf}$ $\lambda(Q) > 0$
such that $P \times Q \subseteq A$.

p-f.

Wlog A is Borel.

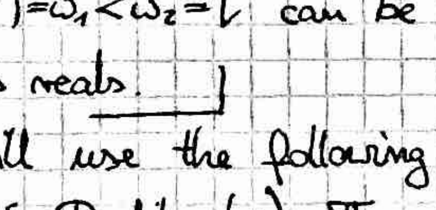
We extend our universe such that $\text{cof}(\mathcal{N}) = \omega_1 < \omega_2 = \mathfrak{c}$

Recall that
$\text{cof}(\mathcal{N}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{N} \wedge \forall I \in \mathcal{N}\ \exists B \in \mathcal{A}\ \ B \supseteq I\}$
minimal size of a base of the null ideal
$\text{cof}(\mathcal{N}) = \omega_1 < \omega_2 = \mathfrak{c}$ can be obtained by adding ω_2 many Sacks reals.

We will use the following result:
(Cichoń, Pawlikowski) There is a dense set in the Boolean algebra $\text{Borel}/\mathcal{N}$ of size $\text{cof}(\mathcal{N})$.

Let $\{B_\alpha : \alpha < \omega_1\} \in \text{Borel} \setminus \mathcal{N}$ be such that
$\forall B \in \text{Borel} \setminus \mathcal{N}\ \exists \alpha < \omega_1\ \ B_\alpha \subseteq B$

By Fubini theorem, there is α such that
$\{x \in [0,1] : B_\alpha \subseteq A_x\}$ has size ω_2



$A_x = \{y : (x,y) \in A\}$

Set $\{x : B_\alpha \subseteq A_x\}$ is coanalytic of size ω_2.
So, it contains a perfect subset.
The sentence $\exists P, Q \in \text{Perf}[0,1]\ (\lambda(Q) > 0 \wedge P \times Q \subseteq A)$
is Σ^1_2-sentence. Hence, it is also true in the ground model

We can prove in a similar way the following version of the previous thm.

Th. Assume $A \subseteq [0,1]^2$ $\lambda^2(A)=1$. Then $\exists P,Q \subseteq [0,1]$
P - Perfect, Q - F_σ, $\lambda(Q)=1$, $P\times Q \subseteq A$.

Th. (Michalski, Ż.) (CH) $\exists L$ - Luzin set $L+L$ is a Bernstein set.

We work in $\mathbb{R}$.
L is a Luzin set means that $\forall M\in \mathcal{M}$ $|L\cap M|\leq \omega$
and $|L| = 2^\omega$
X is a Bernstein set iff X intersects every perfect set but does not contain any perfect set.
$A+B = \{a+b : a\in A, b\in B\}$

The construction goes by transfinite induction.
During the proof we need to show that:
if M_1, M_2 are meager sets, P is a Perfect set.
then there is $x\notin M_1$ s.t $|M_2^c \cap (P-x)| > \omega$.
We can assume that M_1, M_2 are Borel.
Extend our universe such that $cov(\mathcal{M}) = \omega_2$.
In the extension:
Let $A\subseteq P$ be of cardinality ω_1
We will find x s.t. $A \subseteq M_2^c + x$, $x \notin M_1$
It is equivalent to
$$x \in \bigcap_{a\in A} (-M_2^c + a) \cap M_1^c \neq \emptyset$$ (by the def. of $cov(\mathcal{M})$)

The sentence
$$(\exists x)(\exists Q \in Perf)(\forall y)((y\in Q \rightarrow y\in M_2^c \wedge x+y\in P) \wedge x\in M_1^c)$$
is Σ^1_2. So it is true in ground model. $\square$

Th. (Komjáth). Let $(A_n)_{n\in\omega}$ be a sequence of analytic sets such that $\limsup_{n\in X} A_n$ is uncountable for every $X\in[\omega]^\omega$. Then there exists $T\in[\omega]^\omega$ such that $\bigcap_{n\in T} A_n$ is uncountable.

$$t\in \limsup_{n\in X} A_n \iff t\in \bigcap_{n\in\omega}\bigcup_{\substack{k>n\\ k\in X}} A_k \iff$$

$\{n\in X : t\in A_n\}$ is infinite.

The sets $\limsup_{n\in X} A_n$ and $\bigcap_{n\in X} A_n$ are analytic if A_n's are analytic. So uncountable means of size 2^ω

Hints for the proof:

- Extend the universe such that $\mathfrak{t}=\omega_2$.
  $\mathfrak{t}$ is a tower number, i.e. if $\kappa<\mathfrak{t}$ and $\{X_\alpha:\alpha<\kappa\}\subseteq[\omega]^\omega$ s.t. $X_\alpha\subseteq^* X_\beta$ if $\alpha>\beta$ then $\exists X\in[\omega]^\omega$ s.t. $\forall\alpha<\kappa\ X\subseteq^* X_\alpha$.
- Use the decomposition of analytic set into ω_1 many Borel sets to show that the conclusion is Σ^1_2 sentence.