Absoluteness theorems, part 2 (Szymon Żeberski)

Absoluteness of Σ^1_1-sentences.

We say that φ is Σ^1_1-sentence if φ is of the form

$$\exists x \in \mathcal{X} \quad x \in B,$$

where $\mathcal{X}$ is a standard Polish space, $B \in \mathrm{Borel}(\mathcal{X})$.

$\mathcal{X} = \omega^\omega, 2^\omega, [\omega]^\omega, \mathbb{R}, \mathrm{Perf}[0,1]$ etc. and product of such spaces

[We want to understand the space well in different models. Those spaces are usually different sets in different models.]

$B \in \mathrm{Borel}(\mathcal{X})$ means $B = \# b$, where b is a code for B

Borel codes (recepies for Borel sets) $\subseteq \omega^\omega$

Let $\{I_n : n \in \omega\}$ base of $\mathcal{X}$, $I_0 = \emptyset$.

1) Open sets are coded by $b \in \omega^\omega$ s.t. $b(0) = 0$. Then $\# b = \bigcup_{n\in\omega} I_{b(n)}$.

2) Closed sets
$b(0) = 1 \qquad \# b = \left(\bigcup_{n\in\omega} I_{b(n+1)}\right)^c$

3) Countable unions
$b(0) = 2 \qquad$ Let $\varphi : \omega\times\omega \to \omega\setminus\{0\}$ bijection
$\# b = \bigcup_{n\in\omega} \# b_n$, where $b_n(k) = b(\varphi(n,k))$.

4) Complements
$b(0) = 3$
$\# b = (\# b^+)^c$, where $b^+(k) = b(k+1)$.

Th. Σ^1_1-sentences are absolute for standard models of ZF s.t. $\omega_0 + 1 = \{0,1,2,\dots,\omega\} \subseteq \mathbb{M} \subseteq \mathbb{N}$.

p-f.

Let φ be of the form $(\exists x \in \omega^\omega)(x \in A)$ $\quad A \in \Sigma^1_1(\omega^\omega)$

[A code for A is in smaller model $\mathbb{M}$]

$A = \pi[C]$, where $C \subseteq \omega^\omega \times \omega^\omega$ C is closed.

We treat $\omega^\omega \times \omega^\omega$ as $(\omega \times \omega)^\omega$.

$C = [T]$, where T is a tree on $\omega \times \omega$

T is a tree on X iff

$$T \subseteq \bigcup_{n \in \omega} X^n \text{ and } \sigma \in T \rightarrow \sigma \restriction n \in T$$

$$[T] = \{\alpha \in X^\omega : \forall n \ \alpha \restriction n \in T\} \text{ body of } T$$

φ is equivalent to:

$$(\exists x \in \omega^\omega)(\exists y \in \omega^\omega) \ (x,y) \in [T]$$

what means that

T is not well founded

what is absolute. $\square$

Cor. Analytic sets are Lebesgue measurable.

Lemma. $\mathbb{M}$ - model of ZFC. Assume that

$\{x : x \text{ is not a random real over } \mathbb{M}\} \in \mathcal{N}$

Then $\{x : \mathbb{M}[x] \models \varphi(x)\}$ is Lebesgue measurable.

P-f. (of cor.)

Let V be our model of ZFC, a codes our analytic set

Let $M \prec V$ be a countable elementary submodel s.t. $\omega + 1 \subseteq M$, $a \in M$

A formula $\varphi(x) = x \in A$ is Σ^1_1-formula with parameters a, x.

So $M[x] \models \varphi(x) \iff V \models \varphi(x)$

By the lemma we are done. $\square$

Th. (Shoenfield) Assume that $\mathbb{M} \subseteq \mathbb{N}$ are s.t.m. of ZFC and $\omega_1^{\mathbb{N}} \subseteq \mathbb{M}$. Let φ be a Σ^1_2-sentence with parameter from $\mathbb{M}$.

$\varphi = (\exists x \in \omega^\omega)(\forall y \in \omega^\omega)((x,y) \in \# a)$ $\quad$ a is a Borel code $a \in \mathbb{M}$

Then φ is absolute between $\mathbb{M}$ and $\mathbb{N}$.

p-f.

$\varphi \equiv (\exists x \in \omega^\omega)(\neg x \in A)$ $\quad A \in \Sigma^1_1(\omega^\omega)$

$(\exists x \in \omega^\omega) \neg (\exists y \in \omega^\omega)((x,y) \in [T])$,

where T is a tree on $\omega \times \omega$

$(\exists x \in \omega^\omega)$(T_x is well founded), where

$$T_x = \{(y_0, y_1, \dots, y_n) : ((x_0,y_0),(x_1,y_1),\dots,(x_n,y_n)) \in T\}$$

T_x is well-founded

$\equiv$

$\exists \varrho : T_x \to \omega_1$ rank function

[We put here ω_1 instead of ON, becouse rank on a tree T on X has values $< |X|^+$]

We can construct T^* in the following way:

$\sigma \in T^* \equiv \sigma = ((T_0', \varrho_0'), \dots, (T_n', \varrho_n'))$,

where T_i' is finite subtree of T, all its elements "agrees" on the first coordinate

$\varrho_i' : T_i' \to \omega_1$ is a rank function on T_i'.

We also assume that

$T_0' \subseteq T_1' \subseteq \dots \subseteq T_n'$

$\varrho_0' \subseteq \varrho_1' \subseteq \dots \subset \varrho_n'$ i.e $\varrho_1' \restriction T_0' = \varrho_0'$ and so on.

$x_0' \subsetneq x_1' \subsetneq \dots \subsetneq x_n'$

a sequence which is a projection of T_0' on the first coordinate.

ϱ is equivalent to T^* is not well-founded. $\square$

Rm/ We use standard Polish spaces instead of ω^ω in "practice".

Rm. If $\mathbb{N}$ is a generic extension of $\mathbb{M}$, then $ON^{\mathbb{N}} = ON^{\mathbb{M}}$. So, the assumption of Shoenfield thm is satisfied.

Cor. Assume that a measurable cardinal exists. Then Σ^1_2-sets are Lebesgue measurable.

p-f.

We use the same method as in Σ^1_1 case.

The problem: we cannot take countable submodel.

We take:

L - constructible universe

and use lemma:

If a measurable cardinal exists then

$\{x : x \text{ is not a real number over } L\} \in \mathcal{N}$

$\varphi(x) = x \in A$ where A is Σ^1_2

it is Σ^1_2 sentence, so

$L(x) \models \varphi(x) \iff V \models \varphi(x)$.

By the previous lemma $\{x : \varphi(x)\} = A$ is Lebesgue measurable. $\square$