Absoluteness theorems, part 1 (Szymon Żeberski)

$\mathbb{M}, \mathbb{N}$ models, i.e. sets
standard i.e. we have one relation $\in$, So $\mathbb{M} = (\mathbb{M}, \in)$.

$\varphi(x_1, x_2, \dots, x_n)$ formula of language of ZFC
(logical symbols and $\in$ - binary relation symbol)
Formally:

- $x \in y$, $x = y$ are formulas ($=$ is a logical symbol)
- if φ, ψ are formulas, then $\varphi \wedge \psi$, $\varphi \vee \psi, \dots, \neg\varphi$ are formulas
- if φ is a formula, then $\forall x \varphi$, $\exists x \varphi$ are formulas

[Set of formulas is the smallest set closed under $\cdot, \cdot\cdot, \cdot\cdot\cdot$.]

$\varphi(x_1, x_2, \dots, x_n)$ means that $\{x_1, x_2, \dots, x_n\} \supseteq$ free variables of φ.

Def. Assume that $\mathbb{M} \subseteq \mathbb{N}$. We say that $\varphi(x_1, x_2, \dots, x_n)$ is absolute (between $\mathbb{M}$ and $\mathbb{N}$) iff
for every $a_1, a_2, \dots, a_n \in \mathbb{M}$

$$\mathbb{M} \models \varphi(a_1, a_2, \dots, a_n) \iff \mathbb{N} \models \varphi(a_1, a_2, \dots, a_n)$$

[Formally $\varphi(a_1, \dots, a_n)$ is a sentence in the language of ZFC enlarged with new constants $a_1, \dots a_n$.]

Ex. $x \in y$, $x = y$ are absolute.

Ex. $(\exists x)(x \in y)$ is not absolute for
$\mathbb{M} = \{\{\emptyset\}\}$ and $\mathbb{N} = \{\emptyset, \{\emptyset\}\}$.

Def. (of Δ_0 - formulas)

- $x \in y$, $x = y$ are Δ_0 - formulas
- if φ, ψ are Δ_0 - formulas, then $\varphi \wedge \psi$, $\neg\varphi$ are Δ_0 - formulas
- if φ is Δ_0, then $\exists x \in y\ \varphi$, $\forall x \in y\ \varphi$ are Δ_0 - formulas

So Δ_0-formulas are formulas with all quantifiers bounded.

Def. A set A is transitive if for every x

$$x \in A \implies x \subseteq A.$$

Rm. A is transitive iff $y \in x \wedge x \in A \longrightarrow y \in A$.

Fact. If A is transitive then $P(A)$ (the power set of A) and $S(A) = A \cup \{A\}$ (the successor of A) are also transitive.

Ex. $\underset{0}{\underset{\shortparallel}{\emptyset}}, \underset{1}{\underset{\shortparallel}{\{\emptyset\}}}, \underset{2}{\underset{\shortparallel}{\{\emptyset, \{\emptyset\}\}}}, \underset{3}{\underset{\shortparallel}{\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}}}, \dots$

$\omega = \{0, 1, 2, 3, \dots\}$ are transitive sets.

Def. We say that $\mathbb{M}$ is standard transitive model (s.t.m.) if $\mathbb{M} = (\mathbb{M}, \in)$ and $\mathbb{M}$ is transitive.

Fact. Δ_0-formulas are absolute between s.t.m., i.e if $\mathbb{M} \subseteq \mathbb{N}$ are s.t.m. and $\varphi(x_1, x_2, \dots, x_n)$ is Δ_0-formula then for every $a_1, a_2, \dots, a_n \in \mathbb{M}$

$$\mathbb{M} \models \varphi(a_1, \dots, a_n) \iff \mathbb{N} \models \varphi(a_1, \dots, a_n).$$

p-f. (by induction on φ)

$x \in y$, $x = y$ are absolute.

if φ, ψ are absolute then $\varphi \vee \psi, \dots, \neg\varphi$ are absolute.

Assume now that $\varphi(x_0, x_1, \dots, x_n)$ is absolute.

We will show that $\psi(y, x_1, \dots, x_n) = (\exists x_0 \in y)\, \varphi(x_0, x_1, \dots, x_n)$.

Let $b, a_1, \dots, a_n \in \mathbb{M}$

$\mathbb{M} \models \psi(b, a_1, \dots, a_n) \iff \exists a_0 \in \mathbb{M}\ a_0 \in b \wedge \mathbb{M} \models \varphi(a_0, \dots, a_n)$

$\iff \exists a_0 \in \mathbb{N}\ a_0 \in b \wedge \mathbb{N} \models \varphi(a_0, \dots, a_n)$

by absoluteness of φ and transitivity of $\mathbb{M}, \mathbb{N}$ $\square$

Def. We say that φ is Δ_0^{ZF} formula if $ZF \vdash \varphi \Leftrightarrow \psi$ for some ψ Δ_0-formula.

Ex. $\mathrm{Ord}(x)$ = x is an ordinal number is Δ_0^{ZF}.

Indeed

$\mathrm{Ord}(x) \equiv$ x is transitive & $(x, \in)$ is a linear order $\equiv$
$(\forall y \in x)(\forall z \in y)(z \in x) \wedge (\forall y, z \in x)(y = z \vee y \in z \vee z \in y)$.

So, being an ordinal is absolute between s.t.m of ZF.

Def. φ is Σ_1-formula if $\varphi = \exists x \psi$, where ψ is Δ_0-formula
φ is Π_1-formula if $\varphi = \forall x \psi$, —— " ——

We can naturally define Σ_1^{ZF} and Π_1^{ZF}-formulas.

Def. We say that φ is Δ_1^{ZF}-formula if
$ZF \vdash \varphi \Leftrightarrow \psi_1$, $ZF \vdash \varphi \Leftrightarrow \psi_2$, where
ψ_1 is Σ_1-formula and ψ_2 is Π_1-formula.

Fact. If φ is Σ_1-formula and $\mathbb{M} \subseteq \mathbb{N}$ are s.t.m.
then $\mathbb{M} \models \varphi \Longrightarrow \mathbb{N} \models \varphi$.

Fact. Δ_1^{ZF}-formulas are absolute between s.t.m of ZF.

Ex. "$(X, \leq)$ is well-founded order" is absolute between s.t.m. of ZF.

p-f.

Recall that "$(X, \leq)$ is well-founded" means that
$(\forall A)(A \subseteq X \wedge A \neq \emptyset \longrightarrow$ A contains $\leq$-minimal element $)$

So, our formula is clearly Π_1^{ZF}.

Now, notice that every well-founded order has a rank function.

Rank function is $\varrho : x \longrightarrow ON$ (ordinal numbers) satisfies:

$$\varrho(a) \in \varrho(b) \quad \text{for } a, b \in x \text{ s.t. } a < b$$

We can define rank function on x by the following formula:

$$\varrho(b) = \sup\{\varrho(a)+1 \;:\; a < b\}.$$

Notice that

$\exists \varrho \quad \varrho$ is a rank function on x

is Π_1^{ZF} formula. $\square$

Ex. "x is a cardinal number" is Π_1^{ZF}-formula.

We can write it the following way:

$$\mathrm{Ord}(x) \wedge \forall f\, \forall \beta \Big(\underbrace{\mathrm{dom}(f) = \beta,\ \mathrm{rng}(f) \leq \alpha}_{f \text{ is a function}} \longrightarrow$$

$$(\exists y \in x)(y \notin \mathrm{rng}\, f)\Big).$$

This formula is not absolute for s.t.m. of ZFC.