Absoluteness theorems, part 1 (Szymon Żeberski)

$\mathbb{M}, \mathbb{N}$ models, i.e. sets
standard i.e. we have one relation $\in$, So $\mathbb{M} = (\mathbb{M}, \in)$.

$\varphi(x_1, x_2, \dots, x_n)$ formula of language of ZFC
(logical symbols and $\in$ - binary relation symbol)
Formally:

- $x \in y$, $x = y$ are formulas ($=$ is a logical symbol)
- if φ, ψ are formulas, then $\varphi \wedge \psi$, $\varphi \vee \psi, \dots, \neg\varphi$ are formulas
- if φ is a formula, then $\forall x \varphi$, $\exists x \varphi$ are formulas

[Set of formulas is the smallest set closed under $\cdot, \cdot\cdot, \therefore$.]

$\varphi(x_1, x_2, \dots, x_n)$ means that $\{x_1, x_2, \dots, x_n\} \supseteq$ free variables of φ.

Def. Assume that $\mathbb{M} \subseteq \mathbb{N}$. We say that $\varphi(x_1, x_2, \dots, x_n)$ is absolute (between $\mathbb{M}$ and $\mathbb{N}$) iff
for every $a_1, a_2, \dots, a_n \in \mathbb{M}$

$$\mathbb{M} \models \varphi(a_1, a_2, \dots, a_n) \iff \mathbb{N} \models \varphi(a_1, a_2, \dots, a_n)$$

[Formally $\varphi(a_1, \dots, a_n)$ is a sentence in the language of ZFC enlarged with new constants $a_1, \dots a_n$.]

Ex. $x \in y$, $x = y$ are absolute.

Ex. $(\exists x)(x \in y)$ is not absolute for
$\mathbb{M} = \{\{\emptyset\}\}$ and $\mathbb{N} = \{\emptyset, \{\emptyset\}\}$.

Def. (of Δ_0 - formulas)

- $x \in y$, $x = y$ are Δ_0 - formulas
- if φ, ψ are Δ_0 - formulas, then $\varphi \wedge \psi$, $\neg\varphi$ are Δ_0 - formulas
- if φ is Δ_0, then $\exists x \in y\, \varphi$, $\forall x \in y\, \varphi$ are Δ_0 - formulas

So Δ_0-formulas are formulas with all quantifiers bounded.

Def. A set A is transitive if for every x

$$x \in A \implies x \subseteq A.$$

Rm. A is transitive iff $y \in x \wedge x \in A \rightarrow y \in A$.

Fact. If A is transitive then $P(A)$ (the power set of A) and $S(A) = A \cup \{A\}$ (the successor of A) are also transitive.

Ex. $\underset{0}{\overset{\shortparallel}{\emptyset}}, \underset{1}{\overset{\shortparallel}{\{\emptyset\}}}, \underset{2}{\overset{\shortparallel}{\{\emptyset,\{\emptyset\}\}}}, \underset{3}{\overset{\shortparallel}{\{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\}}}, \ldots$

$\omega = \{0,1,2,3,\ldots\}$ are transitive sets.

Def. We say that $\mathbb{M}$ is standard transitive model (s.t.m.) if $\mathbb{M} = (\mathbb{M}, \in)$ and $\mathbb{M}$ is transitive.

Fact. Δ_0-formulas are absolute between s.t.m., i.e if $\mathbb{M} \subseteq \mathbb{N}$ are s.t.m. and $\varphi(x_1, x_2, \ldots, x_n)$ is Δ_0-formula then for every $a_1, a_2, \ldots, a_n \in \mathbb{M}$

$$\mathbb{M} \models \varphi(a_1, \ldots, a_n) \iff \mathbb{N} \models \varphi(a_1, \ldots, a_n).$$

p-f. (by induction on φ)

$x \in y$, $x = y$ are absolute.

if φ, ψ are absolute then $\varphi \vee \psi, \ldots, \neg\varphi$ are absolute.

Assume now that $\varphi(x_0, x_1, \ldots, x_n)$ is absolute.

We will show that $\psi(y, x_1, \ldots, x_n) = (\exists x_0 \in y)\, \varphi(x_0, x_1, \ldots, x_n)$.

Let $b, a_1, \ldots, a_n \in \mathbb{M}$

$\mathbb{M} \models \psi(b, a_1, \ldots, a_n) \iff \exists a_0 \in \mathbb{M}\ a_0 \in b \wedge \mathbb{M} \models \varphi(a_0, \ldots, a_n)$

$\iff \exists a_0 \in \mathbb{N}\ a_0 \in b \wedge \mathbb{N} \models \varphi(a_0, \ldots, a_n)$

by absoluteness of φ and transitivity of $\mathbb{M}, \mathbb{N}$ $\square$

Def. We say that φ is Δ_0^{ZF} formula if $ZF \vdash \varphi \Leftrightarrow \psi$ for some ψ Δ_0-formula.

Ex. $Ord(x)$ = x is an ordinal number is Δ_0^{ZF}.

Indeed

$Ord(x) \equiv$ x is transitive & $(x, \in)$ is a linear order $\equiv$

$(\forall y \in x)(\forall z \in y)(z \in x) \wedge (\forall y, z \in x)(y = z \vee y \in z \vee z \in y)$.

So, being an ordinal is absolute between s.t.m of ZF.

Def. φ is Σ_1-formula if $\varphi = \exists x \psi$, where ψ is Δ_0-formula

φ is Π_1-formula if $\varphi = \forall x \psi$, —— " ——

We can naturally define Σ_1^{ZF} and Π_1^{ZF}-formulas.

Def. We say that φ is Δ_1^{ZF}-formula if

$ZF \vdash \varphi \Leftrightarrow \psi_1$, $ZF \vdash \varphi \Leftrightarrow \psi_2$, where

ψ_1 is Σ_1-formula and ψ_2 is Π_1-formula.

Fact. If φ is Σ_1-formula and $\mathbb{M} \subseteq \mathbb{N}$ are s.t.m.

then $\mathbb{M} \models \varphi \implies \mathbb{N} \models \varphi$.

Fact. Δ_1^{ZF}-formulas are absolute between s.t.m of ZF.

Ex. "$(X, \leq)$ is well-founded order" is absolute between s.t.m. of ZF.

p-f.

Recall that "$(X, \leq)$ is well-founded" means that

$(\forall A)(A \subseteq X \wedge A \neq \emptyset \longrightarrow A$ contains $\leq$-minimal element$)$

So, our formula is clearly Π_1^{ZF}.

Now, notice that every well-founded order has a rank function.

Rank function is $\varrho : x \longrightarrow ON$ (ordinal numbers)

satisfies:

$$\varrho(a) \in \varrho(b) \quad \text{for } a,b \in x \text{ s.t. } a < b$$

We can define rank function on x by the following formula:

$$\varrho(b) = \sup\{\varrho(a)+1 \; : \; a < b\}.$$

Notice that

$\exists \varrho \quad \varrho$ is a rank function on x

is Π_1^{ZF} formula. $\square$

Ex. "x is a cardinal number" is Π_1^{ZF}-formula.

We can write it the following way:

$$\mathrm{Ord}(x) \wedge \forall f \, \forall \beta \Big(\underbrace{\mathrm{dom}(f) = \beta, \; \mathrm{rng}(f) \leq \alpha}_{f \text{ is a function}} \longrightarrow$$

$$(\exists y \in x)(y \notin \mathrm{rng}\, f) \Big).$$

This formula is not absolute for s.t.m. of ZFC.

Absoluteness theorems, part 2 (Szymon Żeberski)

Absoluteness of Σ^1_1-sentences.

We say that φ is Σ^1_1-sentence if φ is of the form

$$\exists x \in \mathcal{X} \quad x \in B,$$

where $\mathcal{X}$ is a standard Polish space, $B \in \text{Borel}(\mathcal{X})$.

$\mathcal{X} = \omega^\omega, 2^\omega, [\omega]^\omega, \mathbb{R}, \text{Perf}[0,1]$ etc. and product of such spaces

[We want to understand the space well in different models. Those spaces are usually different sets in different models.]

$B \in \text{Borel}(\mathcal{X})$ means $B = \# b$, where b is a code for B

Borel codes (recepies for Borel sets) $\subseteq \omega^\omega$

Let $\{I_n : n \in \omega\}$ base of $\mathcal{X}$, $I_0 = \emptyset$.

1) Open sets are coded by $b \in \omega^\omega$ s.t $b(0) = 0$. Then $\# b = \bigcup_{n\in\omega} I_{b(n)}$.

2) Closed sets $b(0) = 1$ $\quad \# b = \left(\bigcup_{n\in\omega} I_{b(n+1)}\right)^c$

3) Countable unions $b(0) = 2$ $\quad$ Let $\varphi : \omega\times\omega \to \omega\setminus\{0\}$ bijection

$\# b = \bigcup_{n\in\omega} \# b_n$, where $b_n(k) = b(\varphi(n,k))$.

4) Complements $b(0) = 3$

$\# b = (\# b^+)^c$, where $b^+(k) = b(k+1)$.

Th. Σ^1_1 - sentences are absolute for standard models of ZF s.t. $\omega_0 + 1 = \{0,1,2,\dots,\omega\} \subseteq \mathbb{M} \subseteq \mathbb{N}$.

p-f.

Let φ be of the form $(\exists x \in \omega^\omega)(x \in A)$ $\quad A \in \Sigma^1_1(\omega^\omega)$

[A code for A is in smaller model $\mathbb{M}$]

$A = \pi[C]$, where $C \subseteq \omega^\omega \times \omega^\omega$ C is closed.

We treat $\omega^\omega \times \omega^\omega$ as $(\omega\times\omega)^\omega$.

$C = [T]$, where T is a tree on $\omega\times\omega$

T is a tree on X iff

$T \subseteq \bigcup_{n\in\omega} X^n$ and $\sigma \in T \to \sigma\restriction n \in T$

$[T] = \{\alpha \in X^\omega : \forall n\ \alpha\restriction n \in T\}$ body of T

φ is equivalent to:

$(\exists x \in \omega^\omega)(\exists y \in \omega^\omega)\ (x,y) \in [T]$

what means that

T is not well founded

what is absolute. $\square$

Cor. Analytic sets are Lebesgue measurable.

Lemma. $\mathbb{M}$ – model of ZFC. Assume that

$\{x : x \text{ is not a random real over } \mathbb{M}\} \in \mathcal{N}$

Then $\{x : \mathbb{M}[x] \models \varphi(x)\}$ is Lebesgue measurable.

P-f. (of cor.)

Let V be our model of ZFC, a codes our analytic set

Let $M \prec V$ be a countable elementary submodel s.t. $\omega+1 \subseteq M$, $a \in M$

A formula $\varphi(x) = x \in A$ is Σ^1_1 – formula with parameters a, x.

So $M[x] \models \varphi(x) \iff V \models \varphi(x)$

By the lemma we are done. $\square$

Th. (Shoenfield) Assume that $\mathbb{M} \subseteq \mathbb{N}$ are s.t.m. of ZFC and $\omega_1^{\mathbb{N}} \leq \mathbb{M}$. Let φ be a Σ^1_2-sentence with parameter from $\mathbb{M}$.

$\varphi = (\exists x \in \omega^\omega)(\forall y \in \omega^\omega)((x,y) \in \# a)$ $\quad$ a is a Borel code $a \in \mathbb{M}$

Then φ is absolute between $\mathbb{M}$ and $\mathbb{N}$.

p-f.

$\varphi \equiv (\exists x \in \omega^\omega)(\neg x \in A) \quad A \in \Sigma^1_1(\omega^\omega)$

$(\exists x \in \omega^\omega) \neg (\exists y \in \omega^\omega)((x,y) \in [T])$,

where T is a tree on $\omega \times \omega$

$(\exists x \in \omega^\omega)(T_x$ is well founded$)$, where

$$T_x = \{(y_0, y_1, \dots, y_n) : ((x_0,y_0),(x_1,y_1),\dots,(x_n,y_n)) \in T\}$$

T_x is well-founded

$\Updownarrow$

$\exists \varrho : T_x \to \omega_1$ rank function

[We put here ω_1 instead of ON, becouse rank on a tree T on X has values $< |X|^+$]

We can construct T^* in the following way:

$\sigma \in T^* \equiv \sigma = ((T_0', \varrho_0'), \dots, (T_n', \varrho_n'))$,

where T_i' is finite subtree of T, all its elements "agrees" on the first coordinate

$\varrho_i' : T_i' \to \omega_1$ is a rank function on T_i'.

We also assume that

$T_0' \leq T_1' \subsetneq \dots \subseteq T_n'$

$\varrho_0' \subseteq \varrho_1' \subsetneq \dots \subset \varrho_n'$ i.e $\varrho_1' \upharpoonright T_0' = \varrho_0'$ and so on.

$x_0' \subsetneq x_1' \subsetneq \dots \subsetneq x_n'$

a sequence which is a projection of T_0' on the first coordinate.

φ is equivalent to T^* is not well-founded. $\square$

Rm/ We use standard Polish spaces instead of ω^ω in "practice".

Rm. If $\mathbb{N}$ is a generic extension of $\mathbb{M}$, then $ON^{\mathbb{N}} = ON^{\mathbb{M}}$. So, the assumption of Shoenfield thm is satisfied.

Cor. Assume that a measurable cardinal exists. Then Σ^1_2-sets are Lebesgue measurable.

p-f.

We use the same method as in Σ^1_1 case.

The problem: we cannot take countable submodel.

We take:

L - constructible universe

and use lemma:

If a measurable cardinal exists then

$\{x : x \text{ is not a real number over } L\} \in \mathcal{N}$

$\varphi(x) = x \in A$ where A is Σ^1_2

it is Σ^1_2 sentence, so

$L(x) \models \varphi(x) \iff V \models \varphi(x)$.

By the previous lemma $\{x : \varphi(x)\} = A$ is Lebesgue measurable. □

Absoluteness theorems, part 3 (Szymon Żeberski)

Applications of Shoenfield theorem

Th. (Mazurkiewicz) Let $f_n : [0,1] \to [0,1]$, $n \in \omega$ be a sequence of Borel functions. Then there exists P a perfect set and $A \in [\omega]^\omega$ such that $\{f_n(x)\}_{n\in A}$ converges for $x \in P$.

p-f.

We enlarge our universe, by a generic extension, to one where $\mathfrak{s} > \omega_1$ (for instance MA)

($\mathfrak{s}$ – splitting number)

$\kappa < \mathfrak{s}$ is equivalent to:

- $\{0,1\}^\kappa$ is sequentially compact
- $(\{0,1\}^\omega)^\kappa$ is sequentially compact
- $[0,1]^\kappa$ is sequentially compact.

In our extension $\omega_1 < \mathfrak{s}$.

Take $X \subseteq [0,1]$ $|X| = \omega_1$

$(f_n \upharpoonright X)_{n\in\omega}$ a sequence of elements in $[0,1]^X$

$[0,1]^X$ is sequentially compact. So

$\exists A \in [\omega]^\omega$ $(f_n \upharpoonright X)_{n \in A}$ is convergent.

The set $\{x : (f_n(x))_{n\in A}$ is convergent$\} =$

$$= \{x : (\forall k)(\exists n)(\forall m, m')(m, m' > n \wedge m, m' \in A \to |f_m(x) - f_{m'}(x)| < \tfrac{1}{k})\}$$

is Borel and uncountabe (contains X). Thus it contains a perfect subset.

We have proved that in extension:

$$\exists P \in Perf([0,1])\ \exists A \in [\omega]^\omega\ \forall x \in [0,1]\ (x \in P \to (f_n(x))_{n \in A} \text{ is convergent})$$

The latter statement is Σ^1_2-sentence.
So, it is true in ground model. $\square$

Th. (Eggelston) Let $A \subseteq [0,1]^2$ be such that $\lambda^2(A) > 0$.
Then there exist $P, Q \subseteq [0,1]$ $P \in \mathrm{Perf}$ $\lambda(Q) > 0$
such that $P \times Q \subseteq A$.

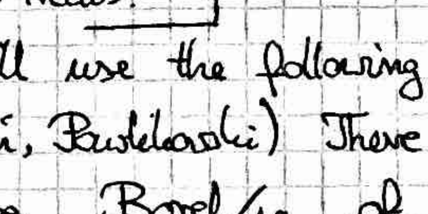
p-f.

Wlog A is Borel.
We extend our universe such that $\mathrm{cof}(\mathcal{N}) = \omega_1 < \omega_2 = \mathfrak{c}$

Recall that
$\mathrm{cof}(\mathcal{N}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{N} \wedge \forall I \in \mathcal{N}\ \exists B \in \mathcal{A}\ \ B \supseteq I\}$
minimal size of a base of the null ideal
$\mathrm{cof}(\mathcal{N}) = \omega_1 < \omega_2 = \mathfrak{c}$ can be obtained by adding ω_2 many Sacks reals.

We will use the following result:
(Cichoń, Pawlikowski) There is a dense set in the Boolean algebra $\mathrm{Borel}/_{\mathcal{N}}$ of size $\mathrm{cof}(\mathcal{N})$.

Let $\{B_\alpha : \alpha < \omega_1\} \in \mathrm{Borel} \setminus \mathcal{N}$ be such that
$\forall B \in \mathrm{Borel} \setminus \mathcal{N}\ \exists \alpha < \omega_1\ \ B_\alpha \subseteq B$
By Fubini theorem, there is α such that
$\{x \in [0,1] : B_\alpha \subseteq A_x\}$ has size ω_2



$A_x = \{y : (x,y) \in A\}$

Set $\{x : B_\alpha \subseteq A_x\}$ is coanalytic of size ω_2.
So, it contains a perfect subset.
The sentence $\exists P, Q \in \mathrm{Perf}[0,1]\ (\lambda(Q) > 0 \wedge P \times Q \subseteq A)$
is Σ^1_2-sentence. Hence, it is also true in the ground model

We can prove in a similar way the following version of the previous thm.

Th. Assume $A \subseteq [0,1]^2$ $\lambda^2(A)=1$. Then $\exists P, Q \subseteq [0,1]$ P - Perfect, Q - F_σ, $\lambda(Q)=1$, $P \times Q \subseteq A$.

Th. (Michalski, Ż.) (CH) $\exists L$ - Luzin set $L+L$ is a Bernstein set.

We work in $\mathbb{R}$.

L is a Luzin set means that $\forall M \in \mathcal{M}$ $|L \cap M| \leq \omega$ and $|L| = 2^\omega$

X is a Bernstein set iff X intersects every perfect set but does not contain any perfect set.

$A+B = \{a+b : a \in A, b \in B\}$

The construction goes by transfinite induction.

During the proof we need to show that:

if M_1, M_2 are meager sets, P is a Perfect set.

then there is $x \notin M_1$ s.t $|M_2^c \cap (P-x)| > \omega$.

We can assume that M_1, M_2 are Borel.

Extend our universe such that $cov(\mathcal{M}) = \omega_2$.

In the extension:

Let $A \subseteq P$ be of cardinality ω_1

We will find x s.t. $A \subseteq M_2^c + x$, $x \notin M_1$

It is equivalent to

$$x \in \bigcap_{a \in A} (-M_2^c + a) \cap M_1^c \neq \emptyset$$ (by the def. of $cov(\mathcal{M})$)

The sentence

$(\exists x)(\exists Q \in Perf)(\forall y)((y \in Q \rightarrow y \in M_2^c \wedge x+y \in P) \wedge x \in M_1^c)$

is Σ^1_2. So it is true in ground model. $\square$

Th. (Komjáth). Let $(A_n)_{n\in\omega}$ be a sequence of analytic sets such that $\limsup_{n\in X} A_n$ is uncountable for every $X\in[\omega]^\omega$.
Then there exists $T\in[\omega]^\omega$ such that $\bigcap_{n\in T} A_n$ is uncountable.

$$t\in \limsup_{n\in X} A_n \iff t\in \bigcap_{n\in\omega}\bigcup_{\substack{k>n\\ k\in X}} A_n \iff$$

$\{n\in X : t\in A_n\}$ is infinite.

The sets $\limsup_{n\in X} A_n$ and $\bigcap_{n\in X} A_n$ are analytic if A_n's are analytic. So uncountable means of size 2^ω

Hints for the proof:

- Extend the universe such that $t=\omega_2$.
  t is a tower number, i.e. if $\kappa<t$ and $\{X_\alpha:\alpha<\kappa\}\subseteq[\omega]^\omega$ s.t. $X_\alpha\subseteq^* X_\beta$ if $\alpha>\beta$ then $\exists X\in[\omega]^\omega$ s.t. $\forall\alpha<\kappa\ X\subseteq^* X_\alpha$.
- Use the decomposition of analytic set into ω_1 many Borel sets to show that the conclusion is Σ^1_2 sentence.