On variants of the extended bicyclic semigroup

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All results are obtained jointly with Oleg Gutik
Definition

A semi(topological) semigroup is a Hausdorff topological space with separately continuous (continuous) semigroup operations.

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A topology $\tau$ on a semigroup $S$ is called:

- *shift-continuous* if $(S, \tau)$ is a semi(topological) semigroup;
- *semigroup* if $(S, \tau)$ is a topological semigroup.
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- \textit{shift-continuous} if \((S, \tau)\) is a semitopological semigroup;
- \textit{semigroup} if \((S, \tau)\) is a topological semigroup.
**Definition (E. Lyapin, 1946)**

The **bicyclic semigroup (monoid)** $C(p, q)$ is the semigroup with the identity $1$ generated by two elements $p$ and $q$ subject only to the condition $pq = 1$.

The distinct elements of the bicyclic monoid are exhibited in the following useful array:

\[
\begin{array}{cccccc}
1 & p & p^2 & p^3 & \ldots \\
q & qp & qp^2 & qp^3 & \ldots \\
q^2 & q^2p & q^2p^2 & q^2p^3 & \ldots \\
q^3 & q^3p & q^3p^2 & q^3p^3 & \ldots \\
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Old classic results on the bicyclic monoid

Theorem (O. Andersen, 1952)
A (0–)simple semigroup with an idempotent is completely (0–)simple if and only if it does not contain the bicyclic semigroup.

Theorem (C. Eberhart, J. Selden, 1969)
The bicyclic semigroup $C(p, q)$ admits only the discrete semigroup Hausdorff topology and if a topological semigroup $S$ contains it as a dense subsemigroup then $C(p, q)$ is an open subset of $S$.

Theorem (M. O. Bertman, T. T. West, 1976)
The bicyclic semigroup $C(p, q)$ admits only the discrete Hausdorff topology $\tau$ such that $(C(p, q), \tau)$ is a semitopological semigroup.
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An interassociate of a semigroup \((S, \cdot)\) is the semigroup \((S, \ast)\) such that for all \(a, b, c \in S\),
\[
a \cdot (b \ast c) = (a \cdot b) \ast c \quad \text{and} \quad a \ast (b \cdot c) = (a \ast b) \cdot c.
\]

Note, that if \(S\) is a monoid, every interassociate must satisfy the condition
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a \ast b = acb
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In particular, if $p$ and $q$ are the generators of the bicyclic semigroup $C(p, q)$ and $m$ and $n$ are fixed nonnegative integers, the operation

$$a *_{m,n} b = aq^m p^n b$$

is known to be an interassociate.

Later for fixed non-negative integers $m$ and $n$ the interassociate $(C(p, q), *_{m,n})$ of the bicyclic monoid $C(p, q)$ will be denoted by $C_{m,n}$.

**Theorem (B. N. Givens, A. Rosin, and K. Linton, 2017)**

For distinct pairs $(m, n)$ and $(s, t)$, the interassociates $(C(p, q), *_{m,n})$ and $(C(p, q), *_{s,t})$ are not isomorphic.

In this paper the bicyclic semigroup $C(p, q)$ and its interassociates are investigated. Also the authors generalized a result regarding homomorphisms on $C(p, q)$ to homomorphisms on its interassociates.
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On the Cartesian product $\mathcal{C}_\mathbb{Z} = \mathbb{Z} \times \mathbb{Z}$ we define the semigroup operation as follows:

$$(a, b) \cdot (c, d) = \begin{cases} 
(a - b + c, d), & \text{if } b < c; \\
(a, d), & \text{if } b = c; \\
(a, d + b - c), & \text{if } b > c,
\end{cases}$$

for $a, b, c, d \in \mathbb{Z}$.

The set $\mathcal{C}_\mathbb{Z}$ with such defined operation we shall call the \textit{extended bicyclic semigroup}.

\textbf{Theorem (I.R. Fihel, O.V. Gutik, 2011)}

Every non-trivial congruence $\mathcal{C}$ on the semigroup $\mathcal{C}_\mathbb{Z}$ is a group congruence, and moreover the quotient semigroup $\mathcal{C}_\mathbb{Z}/\mathcal{C}$ is isomorphic to a cyclic group.

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For arbitrary non-negative integers $m$ and $n$ every Hausdorff topology $\tau$ on $C_{m,n}$ such that $(C_{m,n}, \tau)$ is a semitopological semigroup, is discrete. Thus $C_{m,n}$ is a discrete subspace of any topological semigroup containing it.

Theorem (O.V. Gutik, K.M. Maksymyk, 2016)

If $m$ and $n$ are arbitrary non-negative integers the interassociate $C_{m,n}$ of the bicyclic monoid $C(p,q)$ is a dense subsemigroup of a Hausdorff semitopological semigroup $(S, \cdot)$ and $I = S \setminus C_{m,n} \neq \emptyset$ then $I$ is a two-sided ideal of the semigroup $S$. 
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For arbitrary non-negative integers \( m \) and \( n \) every Hausdorff topology \( \tau \) on \( \mathcal{C}_{m,n} \) such that \((\mathcal{C}_{m,n}, \tau)\) is a semitopological semigroup, is discrete. Thus \( \mathcal{C}_{m,n} \) is a discrete subspace of any topological semigroup containing it.

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Example (O.V. Gutik, K.M. Maksymyk, 2016)

On the semigroup $C_{0}^{m,n}$ we define a topology $\tau_{Ac}$ in the following way:

(i) every element of the semigroup $C_{m,n}$ is an isolated point in the space $(C_{m,n}, \tau_{Ac})$;

(ii) the family $\mathcal{B}(0) = \{ U \subseteq C_{m,n} : U \ni 0 \text{ and } C_{m,n} \setminus U \text{ is finite} \}$ determines a base of the topology $\tau_{Ac}$ at zero $0 \in C_{m,n}$.

i.e., $\tau_{Ac}$ is the topology of the Alexandroff one-point compactification of the discrete space $C_{m,n}$ with the remainder $\{0\}$. The semigroup operation in $(C_{m,n}, \tau_{Ac})$ is separately continuous.

Theorem (O.V. Gutik, K.M. Maksymyk, 2016)

Let $m$ and $n$ be arbitrary non-negative integers. If $C_{0}^{m,n}$ is a Hausdorff locally compact semitopological semigroup, then either $C_{0}^{m,n}$ is discrete or $C_{0}^{m,n}$ is topologically isomorphic to $(C_{m,n}, \tau_{Ac})$. 
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Theorem

For an arbitrary integer $k$ the map $h_k : \mathbb{C}_Z \rightarrow \mathbb{C}_Z$ defined by the formula

$$h_k ((i, j)) = (i + k, j + k),$$

is an automorphism of the extended bicyclic semigroup $\mathbb{C}_Z$ and every automorphism $h : \mathbb{C}_Z \rightarrow \mathbb{C}_Z$ of $\mathbb{C}_Z$ has the form (2). Moreover the group $\text{Aut}(\mathbb{C}_Z)$ of automorphisms of $\mathbb{C}_Z$ is isomorphic to the additive group of integers $\mathbb{Z}(+)$ and this isomorphism $\mathcal{H} : \mathbb{Z}(+) \rightarrow \text{Aut}(\mathbb{C}_Z)$ is defined by the formula $\mathcal{H}(k) = h_k$, $k \in \mathbb{Z}$.

Proposition

Let $m$ and $n$ be arbitrary integers, $(a, b)$ and $(c, d)$ be elements of $\mathbb{C}_Z^{m,n}$. Then the following statements hold.

1. $(a, b) R (c, d) \iff (a = c) \land ((b = d) \lor (b, d \geq m))$.
2. $(a, b) L (c, d) \iff (b = d) \land ((a = c) \lor (a, c \geq n))$.
3. $(a, b) H (c, d) \iff (a, b) = (c, d)$.
4. $(a, b) D (c, d) \iff (a, b) = (c, d) \lor (a, c \geq n) \lor (b, d \geq m)$.
5. $(a, b) J (c, d)$ for all $(a, b), (c, d) \in \mathbb{C}_Z^{m,n}$.
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**Proposition**

Let $m$ and $n$ be arbitrary integers, $(a, b)$ and $(c, d)$ be elements of $\mathcal{C}_\mathbb{Z}^{m,n}$. Then the following statements hold.

1. $(a, b)\mathcal{R}(c, d) \iff (a = c) \land ((b = d) \lor (b, d \geq m))$.
2. $(a, b)\mathcal{L}(c, d) \iff (b = d) \land ((a = c) \lor (a, c \geq n))$.
3. $(a, b)\mathcal{H}(c, d) \iff (a, b) = (c, d)$.
4. $(a, b)\mathcal{D}(c, d) \iff (a, b) = (c, d) \lor (a, c \geq n) \lor (b, d \geq m)$.
5. $(a, b)\mathcal{J}(c, d)$ for all $(a, b), (c, d) \in \mathcal{C}_\mathbb{Z}^{m,n}$.
Theorem

Any two variants of the extended bicyclic semigroup $\mathcal{C}_Z$ are isomorphic.

Theorem

The variant $\mathcal{C}_Z^{0,0}$ of the extended bicyclic semigroup $\mathcal{C}_Z$ is not finitely generated.

Theorem

Let $\tau$ be a Hausdorff shift-continuous topology on the semigroup $\mathcal{C}_Z^{0,0}$. Then every of inequality $a > 0$ or $b > 0$ implies that $(a, b)$ is an isolated point of $(\mathcal{C}_Z^{0,0}, \tau)$.
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Example

We define the topology $\tau^*$ on $\mathcal{C}_Z^{0,0}$ in the following way. Put

(i) $(a, b)$ is an isolated point of $(\mathcal{C}_Z^{0,0}, \tau^*)$ if and only if at least one of the following conditions holds $a > 0$ or $b > 0$;

(ii) if $ab = 0$ and $a + b \leq 0$ we let $A_{(a,b)} = \{(a - i, b - i): i \in \mathbb{N}_0\}$ be any Hausdorff space and $A_{(a,b)}$ is an open-and-closed subset of $(\mathcal{C}_Z^{0,0}, \tau^*)$.

It is obvious that $(\mathcal{C}_Z^{0,0}, \tau^*)$ is a Hausdorff space.

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$(\mathcal{C}_Z^{0,0}, \tau^*)$ is a topological semigroup.
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Thank You for attention!