Can one hear the shape of a group?

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Vyzhnytsya, 2018

(I) History.

Mark Kac (Kremenets, Ternopil region).

1966

Can one hear the shape of a drum?

Phrasing of the title due to Lipman Bers (traced back to Herman Weyl)

$$\begin{cases} \Delta u + \lambda u = 0 \\ u|_{\partial D} = 0 \end{cases}$$

Dirichlet Problem

$$\underbrace{\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots}_{\text{Spectrum}}$$

domain

$\hookrightarrow D \subset \mathbb{R}^2$

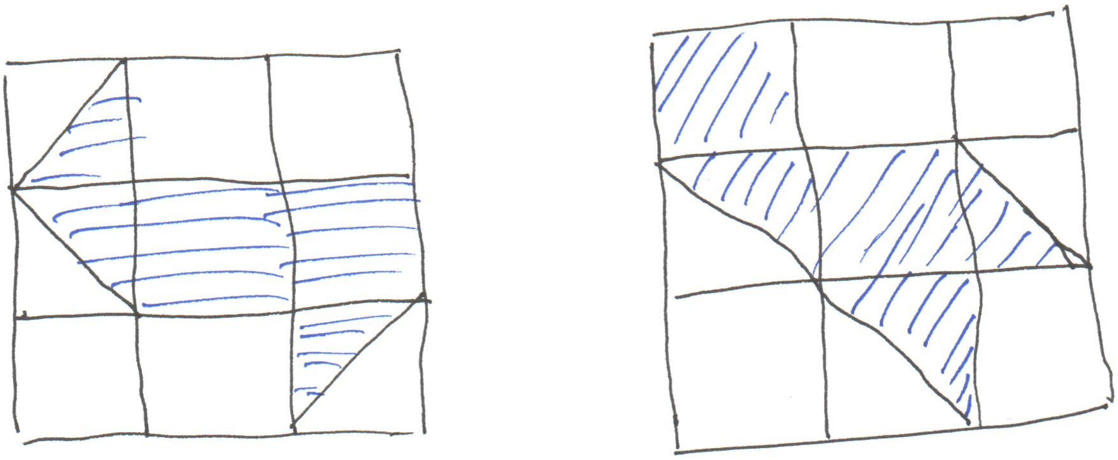
$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Laplace operator

Question. Can the domain Ω be determined (up to isometry) by the spectrum $\{\lambda_n\}_{n=1}^{\infty}$?

Answer NO in general, but YES under extra conditions.

Example D. Webb, C. Gordon, S. Wolpert 1992.



isospectral but not isometric regions.

II Groups as geometric objects

$G = \langle S \rangle$ - finitely generated group

S - finite generating set

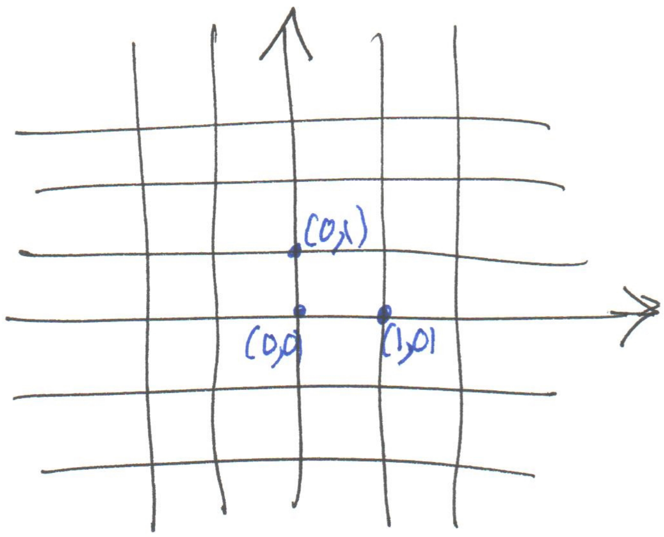
$\Gamma = \Gamma(G, S)$ - Cayley graph

$2 \cdot |S|$ - regular.

$V = G$ - vertices

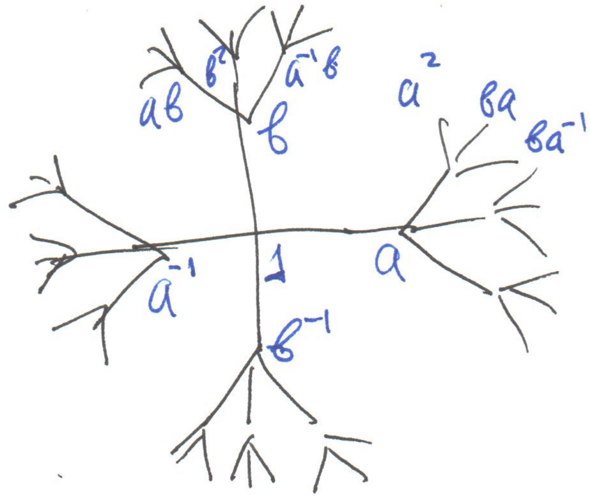
$E = \{(g, sg) \mid g \in G, s \in S\}$ - edges

Γ is $|S|$-regular



$G = \mathbb{Z}^2$

$S = \{(1,0), (0,1)\}$



$G = F_2 = \langle a, b \rangle$

free group

$d(g,h)$ – combinatorial distance

(G, d) – group as a geometric object

Geometric group Theory

Gromov, ...

There are non-isomorphic groups with isometric Cayley graphs.

Generalization. – Schreier graphs

III Spectra of graphs and groups

$\Gamma = (V, E)$ - d-regular graph.

M - Markov operator in $\ell^2(\Gamma)$ $(= \ell^2(V))$

$$(Mf)(x) = \frac{1}{d} \sum_{\substack{y \sim x}} f(y)$$

adjacent

M - self-adjoint. $\|M\| \leq 1$.

$sp(M)$ - spectrum

$sp(M) \subseteq [-1, 1]$

Study of spectra of graphs is a big part of mathematics

μ_v, $v \in V$

spectral measure.

Kadison - Kaplanski Conjecture.

$\Delta = I - M$ - discrete Laplace operator on Γ

I - identity operator

$sp(\Delta) = 1 - sp(M) \subset [0, 2]$.

Γ is amenable $\iff 1 \in Sp(M)$

$\iff 0 \in Sp(\Delta)$.

$\iff \|M\| = 1$.

Def. ~~the~~ ~~The~~ Spectrum of (G, S) is ~~the~~ a ~~the~~ spectrum of the Cayley graph $\Gamma(G, S)$.

Examples: $sp(\mathbb{Z}^2, S) = [-1, 1]$.

$sp(F_2, S) = \left[-\frac{\sqrt{3}}{2}, +\frac{\sqrt{3}}{2}\right]$

$\mathbb{Z}^2$ - amenable

F_2 - non-amenable.

A finitely generated group is amenable if its Cayley graph is amenable.

J. von Neumann, Tarski, Bogolyubov, Banach – Tarski Paradox.

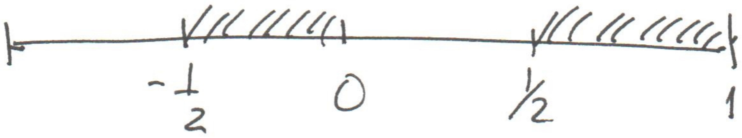
Q. Can one hear the shape of a group?

NO. R. Gri., T. Nagnibeda, A. Pérez
Spectrum of Schreier graphs of spinal groups

Theorem. (Artem Dudko, R. Grigorchuk)

There is a uncountable (of cardinality $2^{\aleph_0}$) family (G_ω, S_ω), $\omega \in \Lambda \subset \{0,1,2\}^{\mathbb{N}}$ of 4-generated amenable groups (of intermediate growth) such that for every $\omega \in \Lambda$

$$sp(G_\omega, S'_\omega) = [-\tfrac{1}{2}, 0] \cup [\tfrac{1}{2}, 1]$$



But $\Gamma_\omega = \Gamma_{\cancel{\omega}}(G_\omega, S_\omega)$ are not only not isometric but even are pairwise not quasi-isometric.

Def. (X, d_1), (Y, d_2) are quasi-isometric if there are maps

$$\varphi : X \longrightarrow Y$$

$$\psi : Y \longrightarrow X$$

and constants A, B, C such that

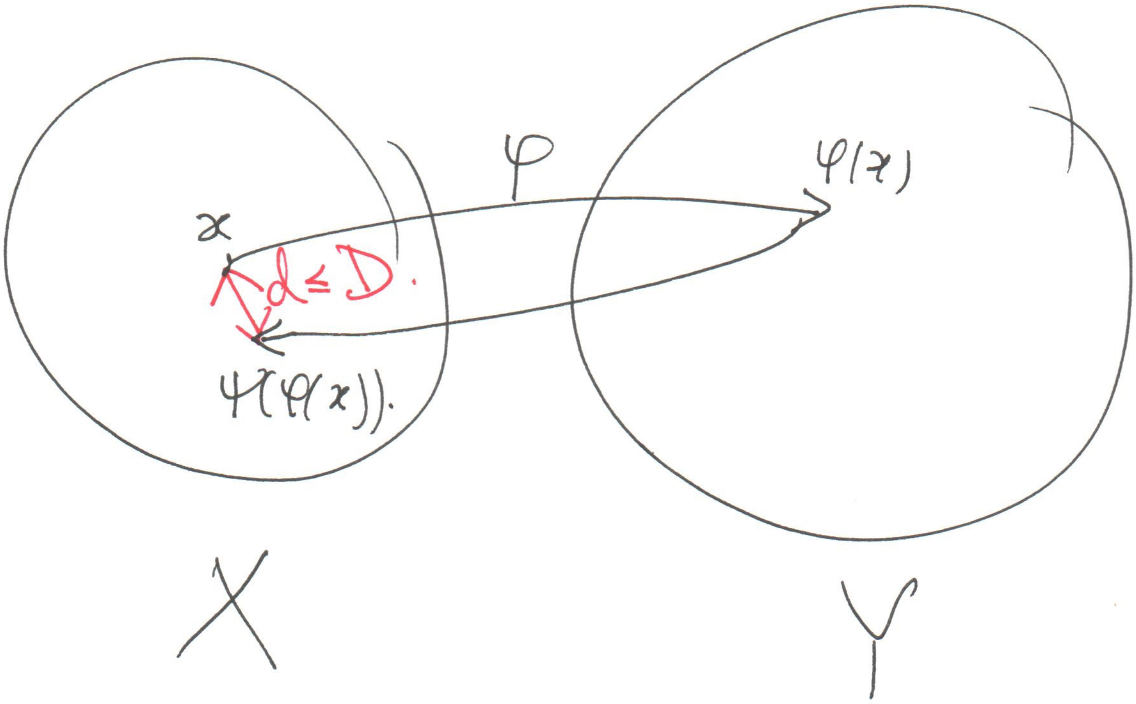
a) $$d_2(\varphi(x_1), \varphi(x_2)) \leq A\, d_1(x_1, x_2) + B$$

$\forall \; x_1, x_2 \in X$

b) $\psi \circ \varphi : X \longrightarrow X$

is on a bounded distance from the identity map.

$$\forall x \in X, \quad d_1(x, \psi(\varphi(x))) \leq D.$$



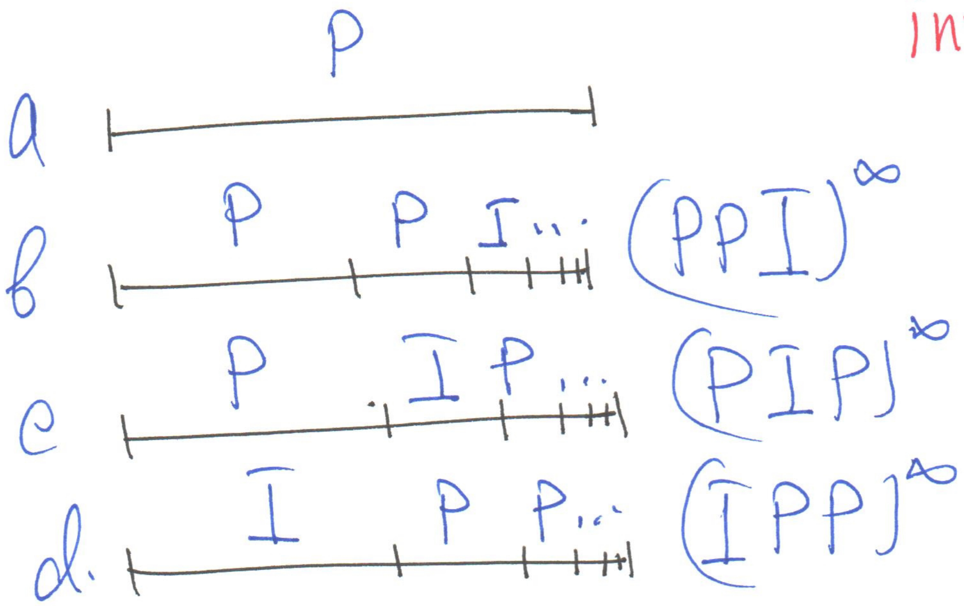
c) $\forall y \in Y \quad \exists x \in X$ s.t.

$$d(y, \varphi(x)) \leq D.$$

The groups.

The First example

$$G = \langle a, b, c, d \rangle$$



involutions
$a^2 = b^2 = c^2 = d^2 = 1$,

permutation of halves

I - identity map.

(Burnside, 1902).

Burnside type group
intermediate growth (between polynomial and exponential) J. Milnor. 1968.

Amenable but not elementary amenable (M. Day 1957).

Just - infinite

Branch, ...

$$G = G_{(012)^\infty}$$

$$G_\omega = \langle \underbrace{a, b_\omega, c_\omega, d_\omega}_{S_\omega} \rangle$$

$$\omega \in \{0,1,2\}^{\mathbb{N}} = \Omega$$

$\Lambda = \{\omega \mid$ each symbol 0, 1, 2 occure ∞ many times$\}$

$$0 \longleftrightarrow \begin{pmatrix} P \\ P \\ I \end{pmatrix}, \quad 1 \longleftrightarrow \begin{pmatrix} P \\ I \\ P \end{pmatrix}, \quad 2 \longleftrightarrow \begin{pmatrix} I \\ P \\ P \end{pmatrix}.$$

$$\Omega \ni \omega \longleftrightarrow \begin{pmatrix} U_\omega \\ V_\omega \\ W_\omega \end{pmatrix}$$

$3 \times \infty$ matrix.

$$U_\omega,\ V_\omega,\ W_\omega \in \{P, I\}^{\mathbb{N}}.$$

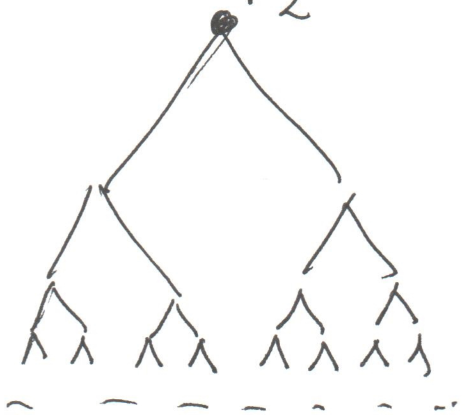
$$\downarrow \quad\quad \downarrow \quad\quad \downarrow$$

$$b_\omega \quad\quad c_\omega \quad\quad d_\omega.$$

determine.

$$G_\omega \leq \mathrm{Aut}(T_2)$$

T_2



T_2 – binary rooted tree.

$$G = \langle a, b, c, d \mid 1 = a^2 = b^2 = c^2 = d^2 = bcd = \sigma^k((ad)^4) = \sigma^k((adacac)^4),\ k = 0, 1, 2, \dots \rangle$$

I. Lysenok.

Y. Muntyan.

$$\sigma: \begin{cases} a \to aca \\ b \to d \\ c \to b \\ d \to c. \end{cases}$$

– substitution.

aperiodic order.

Gr., D. Lenz, T. Nagnibeda

Theorem (A. Dudko, R. Gr...). For every $\omega \in \Lambda$ there exists a $2^{\aleph_0}$ of amenable groups covering the group G_ω with the same spectrum

$$[-\tfrac{1}{2}, 0] \cup [\tfrac{1}{2}, 1].$$

$$\begin{array}{c} G \\ \downarrow \\ G_\omega \end{array}$$

covering

epimorphism

Hulanicki type theorem for graphs.

Theorem (Hulanicki). TFAE

- G - is amenable.
- $1_G \prec \lambda_G$
- $\forall \rho$ - unitary representation $\rho \prec \lambda_G$

1_G - trivial repres.

λ_G - regular repres.

weak containment

$\Gamma = (V, E, w)$ - weighted graph

w — weight

$w : E \to \mathbb{C}$

M_w - weighted Markov operator

Δ_w - weighted Laplacian

Theorem (A. Dudko, R. Gr).

(Weak Hulanicki Theorem for graphs).

Let Γ_1 be a connected weighted graph of uniformly bounded degree which covers a weighted graph Γ_2. such that either

a) Γ_1 is amenable and Γ_2 is finite

or

b) Γ_1 has subexponential growth

Then $\quad sp(\Gamma_2) \subset sp(\Gamma_1)$.

Question. Let Γ_1 be amenable and

$\Gamma_1 \twoheadrightarrow \Gamma_2$. Does $sp(\Gamma_2) \subset sp(\Gamma_1)$?

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