Compatible group topologies and the Mackey topology

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Topological groups

Definition
A topological group is a pair \((G, \tau)\), where \((G, \cdot)\) is a group and \(\tau\) is a topology on \(G\) such that the group operations \(\cdot : G \times G \to G\) and \(\text{inv} : G \to G, \ x \mapsto x^{-1}\) are continuous. A topological isomorphism is an isomorphism of groups which is also a homeomorphism.
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2. \(\mathbb{R}\) and more generally every topological vector space is a topological group.
3. \(\text{GL}(n, \mathbb{R})\) is a non–abelian topological group.
Proposition

1. Let \((G_i)_{i \in I}\) (\(I \neq \emptyset\)) a family of topological group. Then \(\prod_{i \in I} G_i\) with the product topology is a topological group.
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   \(\prod_{i \in I} G_i\) with the product topology is a topological group.

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3. Let \(H\) be a normal subgroup of \(G\). If \(G/H\) endowed with 
   the quotient topology \((O \subseteq G/H\) is open iff 
   \(\{x \in G : x + H \in O\}\) is open) then \(G/H\) is a topological 
   group. It is Hausdorff iff \(H\) is a closed normal subgroup.

Example

\(T := \mathbb{R}/\mathbb{Z}\) is a topological group. It is topologically isomorphic to 
\(\{z \in \mathbb{C} : |z| = 1\}\).

Sometimes we identify \(T\) with \([-\frac{1}{2}, \frac{1}{2}]\).
Proposition

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Character Group

Definition
Let \((G, \tau)\) be a topological group. The set \(\{\chi : G \to \mathbb{T} : \chi \text{ is a continuous homomorphisms}\}\) is called character group.
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Examples
The following mappings are topological isomorphisms:

1. $\mathbb{R} \to \mathbb{R}^\wedge$, $x \mapsto \chi_x$ where $\chi_x : \mathbb{R} \to \mathbb{T}$, $t \mapsto 2\pi xt + \mathbb{Z}$,
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2. \(\mathbb{Z}^\wedge \to \mathbb{T}, \ \chi \mapsto \chi(1),\) and
3. \(\mathbb{Z}_p \to \mathbb{Z}(\mathbb{P}^{\infty})^\wedge, \ \sum_{n \geq 0} k_n p^n \mapsto (\frac{a}{p^m} \mapsto \sum_{n < m} ak_n p^{n-m} + \mathbb{Z}).\)
Polars

Definition
Let $T_+ := \{x + \mathbb{Z} : |x| \leq \frac{1}{4}\}$.

Remark
We have $A^\perp \subseteq A^{\triangle}$.

$A^\perp$ is a subgroup.
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$$A^\triangleright = \{\chi \in G^\wedge : \chi(A) \subseteq T_+\}$$

is called the polar of $A$.

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We have $A^\perp \subseteq A^\triangleright$. $A^\perp$ is a subgroup.
Definition (Vilenkin)

A subset $A$ of a Hausdorff group $G$ is called quasi–convex, iff $A = (A^>)^<$ holds.
Locally quasi–convex groups

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A subset $A$ of a Hausdorff group $G$ is called **quasi–convex**, iff $A = (A^>)^<$ holds. If a Hausdorff group $G$ has a neighborhood basis at 0 consisting of quasi–convex sets, then $G$ is called a **locally quasi–convex group** (**lqc group**).
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Examples

1. Every character group with the compact–open topology is lqc.
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**Locally quasi–convex groups**

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1. Every character group with the compact–open topology is lqc.
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3. Subgroups and products of lqc groups are lqc.
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Examples

1. Every character group with the compact–open topology is lqc.
2. Every locally compact abelian group is lqc.
3. Subgroups and products of lqc groups are lqc.
4. In general, quotient groups are not lqc.
Locally quasi–convex groups

Remark

*Every lqc group is MAP (i.e. the characters separate the points).*
Locally quasi–convex groups

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Every lqc group is MAP (i.e. the characters separate the points).

Remark (Banaszczyk)
A Hausdorff topological vector space is locally convex iff it is a lqc group.
Compact polars

Notation

Let $G$ be a MAP group. The topology on $G^\wedge$ induced by

$$ G^\wedge \to T^G, \, \chi \mapsto (\chi(x))_{x \in G} $$

is denoted by $\sigma(G^\wedge, G)$. 
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Proposition (Arzela-Ascoli)

Let \((G, \tau)\) be an abelian topological group. Let \( U \subseteq G \) be a neighborhood of 0. Then the polar \( U^\triangleright \) is compact in the compact open topology of \( G^\wedge \) and hence compact in the topology \( \sigma(G^\wedge, G) \). So both topologies coincide on \( U^\triangleright \).
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Compatible topologies

Definition
Let \((G, \tau)\) be a lqc group. A lqc group topology \(\tilde{\tau}\) is called **compatible** with \(\tau\) if (algebraically) \((G, \tau) = (G, \tilde{\tau})\) holds.

Remark 1. \(C(G, \tau)\) is a poset (partially ordered set).

Remark 2. Let \((G, \tau)\) be a lqc group. \(\sigma(G, \sigma(G) = \text{bottom element of } C(G, \tau)\).
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Let \((G, \tau)\) be a lqc group. A lqc group topology \(\tilde{\tau}\) is called **compatible** with \(\tau\) if (algebraically) \((G, \tau)^\wedge = (G, \tilde{\tau})^\wedge\) holds. We denote by \(C(G, \tau)\) or \(C(G, G^\wedge)\) the set of all lqc group topologies on \(G\) which are compatible with \(\tau\).

**Remark**

1. \(C(G, \tau)\) is a poset (partially ordered set).
2. Let \((G, \tau)\) be a lqc group. \(\sigma(G, G^\wedge)\) is the bottom element in \(C(G, \tau)\).
Mackey topology

Definition
In case $\mathcal{C}(G, \tau)$ has a top element $\mu$, this topology is called the Mackey topology on $(G, \tau)$. 
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Definition
In case $\mathcal{C}(G, \tau)$ has a top element $\mu$, this topology is called the **Mackey** topology on $(G, \tau)$.

Question
1. *Does the Mackey topology always exist?*
Mackey topology

Definition
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Question
1. Does the Mackey topology always exist?
2. Which topologies are the Mackey topology?
Lemma (Chasco, Martín Peinador, Tarieladze; 1995)

Let $G$ be a Baire group and suppose that $(\chi_n)$ is a sequence in $G^\wedge$ which is pointwise a Cauchy sequence (i.e. $(\chi_n(x))$ is a Cauchy sequence for every $x \in G$). Then $\{\chi_n : n \in \mathbb{N}\}$ is equicontinuous (i.e. for every neighborhood $V$ of $0$ in $\mathbb{T}$ there exists a neighborhood $U$ of $0$ in $G$ such that $\chi_n(U) \subseteq W$).
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Theorem (Chasco, Martín Peinador, Tarieladze; 1995)

Let $(G, \tau)$ be a completely metrizable lqc group. Then $\tau$ is the top element in $\mathcal{C}(G, \tau)$, i.e. $\tau$ is the Mackey topology.
Mackey topology

Theorem (Troallic; Chasco, Martín Peinador; 1995)

Every compact group is Mackey.

Remark
If $G$ is a (pseudo) compact group, then $C(G)$ is a singleton.

Is it possible to characterize those precompact groups which are Mackey?
Mackey topology

Theorem (Troallic; Chasco, Martín Peinador; 1995)
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Remark
If $G$ is a (pseudo) compact group, then $C(G)$ is a singleton. Is it possible to characterize those precompact groups which are Mackey?
Maximal and top elements

Remark

1. **By a result of Chasco, Martín Peinador and Tarieladze, the supremum of a totally ordered set of compatible topologies is again compatible. So, by Zorn’s Lemma, the poset $\mathcal{C}(G)$ has maximal elements.**

2. **Gabriyelyan:** *How can the maximal elements in $\mathcal{C}(G)$ be described?*

3. **It is straightforward to prove that the poset $\mathcal{C}(G)$ is a lattice iff it has a top element.*
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Finite products and open subgroups

All groups are assumed to be lqc and abelian.

**Proposition (Martín Peinador, Dikranjan, L.A.)**

Let $(G_1, \tau_1)$ and $(G_2, \tau_2)$ be lqc groups. Then

$C(G_1, \tau_1) \times C(G_2, \tau_2) \to C(G_1 \times G_2, \tau_1 \times \tau_2), \ (\eta_1, \eta_2) \mapsto \eta_1 \times \eta_2$

is a poset embedding.
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Proposition (Martín Peinador, Dikranjan, L.A.)
Let \((G, \tau)\) be a lqc group and \(H\) an open subgroup. Then there exists a canonical poset embedding

\[ \psi : C(H, \tau|_H) \longrightarrow C(G, \tau). \]
Notation

For an arbitrary MAP group \((G, \tau)\), we denote by \(\tau_{lqc}\) the finest locally quasi–convex group topology which is coarser than \(\tau\). A neighborhood basis given by the sets \((q_c(U)) = (U^{>\triangleleft})_U\) where \(U\) runs through all neighborhoods of 0 in \(\tau\).
Quotient groups

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A neighborhood basis given by the sets \((qc(U)) = (U \triangledown \triangleleft) \cup\) where \(U\) runs through all neighborhoods of 0 in \(\tau\).

Let \(H\) be a closed subgroup of the topological abelian group \((G, \tau)\). Denote by \(q : G \to G/H\) the canonical projection and by and by \(Q(\tau)\) the quotient topology of \(\tau\) on \(G/H\).
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If \(Q(\tau)\) is MAP, we denote by \(Q_{lqc}(\tau)\) the topology \((Q(\tau))_{lqc}\).
Quotient groups

Theorem (Diaz Nieto, Dikranjan, Martín Peinador, L.A.; 2015)

Let $H$ be a closed subgroup of $G$ such that $G/H$ is a MAP group.

$$Q_{lqc} : \mathcal{C}(G, \tau) \longrightarrow \mathcal{C}(G/H, Q_{lqc}(\tau)), \quad \tau \longmapsto Q_{lqc}(\tau),$$

is an order-preserving mapping.

Let $H$ be a subgroup of the lqc Hausdorff group $(G, \tau)$ such that $(G/H, Q(\tau))$ is lqc. The mapping

$$\Theta : \mathcal{C}(G/H, Q(\tau)) \longrightarrow \mathcal{C}(G, \tau), \quad \emptyset \longmapsto q^{-1}(\emptyset) \vee \sigma(G, G^\wedge),$$

is a poset embedding with left inverse $Q_{lqc}$. 

Proposition (Diaz Nieto, Martín Peinador)

Let $H$ be a subgroup of the locally quasi-convex Hausdorff group $(G, \tau)$ such that $(G/H, Q(\tau))$ is lqc. If $\tau$ is the Mackey topology on $G$, then $Q(\tau)$ is the Mackey topology on $G/H$. 
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Let $H$ be a subgroup of the locally quasi-convex Hausdorff group $(G, \tau)$ such that $(G/H, Q(\tau))$ is lqc. If $\tau$ is the Mackey topology on $G$, then $Q(\tau)$ is the Mackey topology on $G/H$.

Theorem (L.A.)
Let $G$ be a lqc group. Let $K$ be a compact subgroup. Then $G/K$ is again lqc and $\Theta : \mathcal{C}(G) \rightarrow \mathcal{C}(G/K)$ is an isomorphism of posets.
LCA groups

Definition
A topological space is said to be Čech–complete, if it is a $G_δ$ subset of a compact space. In particular, every locally compact space is Čech–complete.
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**Proposition**

*An abelian topological group $G$ is a Čech–complete space iff it has a compact subgroup $K$ such that $G/K$ is completely metrizable.*
LCA groups

Definition
A topological space is said to be Cech–complete, if it is a $G_\delta$ subset of a compact space. In particular, every locally compact space is Cech–complete.

Proposition
An abelian topological group $G$ is a Cech–complete space iff it has a compact subgroup $K$ such that $G/K$ is completely metrizable.

Theorem
Let $(G, \tau)$ be Cech complete lqc group. e.g. a locally compact abelian (LCA) group. Then $\tau$ is the top element in $C(G, \tau)$. 
Remark

Let $G$ be a LCA group. What can be said about the lattice $\mathcal{C}(G)$?
Open Questions

Remark

1. *It $\tau$ is the Mackey topology on $G$ and $H$ is an open subgroup of $(G,\tau)$, then $\tau|_H$ is the Mackey topology on $H$.**
Open Questions

Remark

1. If \( \tau \) is the Mackey topology on \( G \) and \( H \) is an open subgroup of \( (G, \tau) \), then \( \tau|_H \) is the Mackey topology on \( H \).

2. Let \( (G, \tau) \) be a lqc group. Let \( H \) be an open subgroup. Assume that \( (H, \tau|_H) \) is the Mackey topology on \( H \). It is an open question whether \( \tau \) is the Mackey topology on \( G \).
Remark

1. It $\tau$ is the Mackey topology on $G$ and $H$ is an open subgroup of $(G, \tau)$, then $\tau|_H$ is the Mackey topology on $H$.

2. Let $(G, \tau)$ be a lqc group. Let $H$ be an open subgroup. Assume that $(H, \tau|_H)$ is the Mackey topology on $H$. It is an open question whether $\tau$ is the Mackey topology on $G$.

3. Let $\tau_i$ be Mackey topologies on $G_i$ for $i \in \{1, 2\}$. It is also an open question whether the product topology $\tau_1 \times \tau_2$ is the Mackey topology on $G_1 \times G_2$. 
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**Definition**

Let \((G, \tau)\) be a lqc group. A lqc group topology \(\tilde{\tau}\) is called **compatible** with \(\tau\) if (algebraically) \((G, \tau)^\wedge = (G, \tilde{\tau})^\wedge\) holds. We denote by \(\mathcal{C}(G, \tau)\) or \(\mathcal{C}(G, G^\wedge)\) the set of all lqc group topologies on \(G\) which are compatible with \(\tau\).

**Theorem**

Let \((G, \tau)\) be a LCA (locally compact abelian) group. Then \(\tau\) is the top element in \(\mathcal{C}(G, \tau)\) and \(\sigma(G, G^\wedge)\) is the bottom element.
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The Question

Question

Let $G$ be a LCA group. Does $|\mathcal{C}(G)| \leq 2$ hold if and only if $G$ is compact?

What can be said about the size of $\mathcal{C}(G)$, the width of $\mathcal{C}(G)$, the length of chains in $\mathcal{C}(G)$?
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Let $G$ be a LCA group. Does $|C(G)| \leq 2$ hold if and only if $G$ is compact?

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Compatible group topologies and the Mackey topology

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The Question

**Question**

Let $G$ be a LCA group. Does $|\mathcal{C}(G)| \leq 2$ hold if and only if $G$ is compact?

What can be said about the size of $\mathcal{C}(G)$, the width of $\mathcal{C}(G)$, the length of chains in $\mathcal{C}(G)$?

A **chain** is a totally ordered subset of $\mathcal{C}(G)$.

The **width** of a poset is the maximal size of a set of pairwise incomparable elements.
Theorem

Let \((G, \tau)\) be a LCA group. Then \(\tau\) is the Mackey topology.
Cardinal estimates

Theorem
Let \((G, \tau)\) be a LCA group. Then \(\tau\) is the Mackey topology.

Proposition
Let \((G, \tau)\) be a LCA group. Then \(|\mathcal{C}(G, \tau)| \leq 2|\tau| \leq 2^{w(G)}\) where \(w(G)\) denotes the weight of \((G, \tau)\). In particular:

1. If \(G\) is discrete, then \(|\mathcal{C}(G)| \leq 2^{|G|}\).
Cardinal estimates

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Let \((G, \tau)\) be a LCA group. Then \(\tau\) is the Mackey topology.

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1. If \(G\) is discrete, then \(|\mathcal{C}(G)| \leq 2^{2^{|G|}}\).

2. \(|\mathcal{C}(\mathbb{Z})| \leq 2^{c}, |\mathcal{C}(\mathbb{Z}(p^\infty))| \leq 2^{c}\)
Cardinal estimates

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Let \((G, \tau)\) be a LCA group. Then \(\tau\) is the Mackey topology.

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Let \((G, \tau)\) be a LCA group. Then \(|C(G, \tau)| \leq 2^{\tau} \leq 2^{2^w(G)}\) where \(w(G)\) denotes the weight of \((G, \tau)\). In particular:

1. If \(G\) is discrete, then \(|C(G)| \leq 2^{2^{|G|}}\).
2. \(|C(\mathbb{Z})| \leq 2^c, |C(\mathbb{Z}(p^\infty))| \leq 2^c\)
3. \(|C(\mathbb{R})| \leq 2^c\)
Cardinal estimates

**Theorem**

Let \((G, \tau)\) be a LCA group. Then \(\tau\) is the Mackey topology.

**Proposition**

Let \((G, \tau)\) be a LCA group. Then \(|\mathcal{C}(G, \tau)| \leq 2^{\tau} \leq 2^{2^w(G)}\) where \(w(G)\) denotes the weight of \((G, \tau)\). In particular:

1. If \(G\) is discrete, then \(|\mathcal{C}(G)| \leq 2^{2^{|G|}}\).
2. \(|\mathcal{C}(\mathbb{Z})| \leq 2^c\), \(|\mathcal{C}(\mathbb{Z}(p^\infty))| \leq 2^c\)
3. \(|\mathcal{C}(\mathbb{R})| \leq 2^c\)

**Proof.**

Assume that \(\eta\) is compatible with \(\tau\). Then \(\eta \subseteq \tau\). Hence \(\mathcal{C}(G, \tau) \subseteq \mathcal{P}(\tau)\). \(\square\)
Theorem
Structure theorem for LCA groups
Let $G$ be a LCA group. There exists a unique $n \in \mathbb{N}_0$ and a LCA group $H$ which has a compact open subgroup $K$ such that $G$ is topologically isomorphic to $\mathbb{R}^n \times H$. 
Theorem

Structure theorem for LCA groups
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Theorem

Let $(G, \tau)$ be a lqc group and let $K \leq G$ be a compact subgroup. Then $\mathcal{C}(G, \tau)$ and $\mathcal{C}(G/K, Q(\tau))$ are isomorphic posets.
Theorem

Structure theorem for LCA groups
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Theorem

Let $(G, \tau)$ be a lqc group and let $K \leq G$ be a compact subgroup. Then $\mathcal{C}(G, \tau)$ and $\mathcal{C}(G/K, Q(\tau))$ are isomorphic posets.

Corollary

Let $G = \mathbb{R}^n \times H$ be a LCA group. Let $K$ be a compact open subgroup of $H$. Then $\mathcal{C}(G)$ is isomorphic to $\mathcal{C}(\mathbb{R}^n \times H/K)$. 

Compatible group topologies and the Mackey topology
Proposition

Let \((G_1, \tau_1)\) and \((G_2, \tau_2)\) be lqc groups. Then

\[
\mathcal{C}(G_1, \tau_1) \times \mathcal{C}(G_2, \tau_2) \to \mathcal{C}(G_1 \times G_2, \tau_1 \times \tau_2), \quad (\eta_1, \eta_2) \mapsto \eta_1 \times \eta_2
\]

is a poset embedding.
Proposition

Let \((G_1, \tau_1)\) and \((G_2, \tau_2)\) be lqc groups. Then

\[ C(G_1, \tau_1) \times C(G_2, \tau_2) \to C(G_1 \times G_2, \tau_1 \times \tau_2), \ (\eta_1, \eta_2) \mapsto \eta_1 \times \eta_2 \]

is a poset embedding.

In order to study \(C(G)\) we have to study \(C(\mathbb{R})\) and \(C(D)\) where \(D\) is an infinite discrete group.
Reduction to infinite sums

Let $D$ be a discrete group of infinite rank. Then $D$ has an (open) subgroup

$$D_0 = \sum_{i \in I} D_i$$

where each $D_i$ is a finite or countable subgroup and

$$|I| = \text{rank}(D) = |D|.$$
Reduction to infinite sums

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Proposition

Let $H$ be an open subgroup of the lqc group $G$. Then

$C(H) \to C(G)$, $\eta \mapsto \eta_0$ where $\eta_0$ is the group topology on $G$ which satisfies $\eta_0|H = \eta$ and makes $H$ an open subgroup of $G$, is a poset embedding.
Reduction to infinite sums

Let $D$ be a discrete group of infinite rank. Then $D$ has an (open) subgroup

$$D_0 = \sum_{i \in I} D_i$$

where each $D_i$ is a finite or countable subgroup and

$$|I| = \text{rank}(D) = |D|.$$

**Proposition**

Let $H$ be an open subgroup of the lqc group $G$. Then

$$\mathcal{C}(H) \to \mathcal{C}(G), \ \eta \mapsto \eta_0$$

where $\eta_0$ is the group topology on $G$ which satisfies $\eta_0|H = \eta$ and makes $H$ an open subgroup of $G$, is a poset embedding.

**Corollary**

$$\mathcal{C}(D_0) \text{ embeds in } \mathcal{C}(D).$$
Notation

A free filter on a set $X \neq \emptyset$ is a filter which does not converge. I.e. the intersection of all members of the filter is empty. Denote by

$$\mathcal{F}(X)$$

the set of all free filters on $X$. 
Notation

A free filter on a set $X \neq \emptyset$ is a filter which does not converge. I.e. the intersection of all members of the filter is empty.

Denote by $\mathcal{F}(X)$ the set of all free filters on $X$.

Let $\delta$ denote the discrete topology on $D_0$. 

Compatible group topologies and the Mackey topology

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Proposition

Let $\mathcal{F}$ be a free filter on $I$. For $F \in \mathcal{F}$, denote by

$$D_F := \sum_{i \in F} D_i \leq D_0.$$

$(D_F)_{F \in \mathcal{F}}$ forms a neighborhood basis at 0 of a lqc group topology $\tau_{\mathcal{F}}$ on $D_0$. 
Proposition

Let $\mathcal{F}$ be a free filter on $I$. For $F \in \mathcal{F}$, denote by

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$(D_F)_{F \in \mathcal{F}}$ forms a neighborhood basis at $0$ of a lqc group topology $\tau_{\mathcal{F}}$ on $D_0$.

$$\sigma(D_0, D_0^\wedge) < \tau_{\mathcal{F}} \vee \sigma(D_0, D_0^\wedge) < \delta.$$
Proposition

Let $\mathcal{F}$ be a free filter on $I$. For $F \in \mathcal{F}$, denote by

$$D_F := \sum_{i \in F} D_i \leq D_0.$$ 

$(D_F)_{F \in \mathcal{F}}$ forms a neighborhood basis at $0$ of a lqc group topology $\tau_\mathcal{F}$ on $D_0$.

$$\sigma(D_0, D_0^\wedge) < \tau_\mathcal{F} \lor \sigma(D_0, D_0^\wedge) < \delta.$$ 

Moreover,

$$\mathfrak{g}(I) \rightarrow \mathcal{C}(D_0), \quad \mathcal{F} \mapsto \tau_\mathcal{F} \lor \sigma(D_0, D_0^\wedge)$$

is a poset embedding.
Corollary

$$2^{2^{|I|}} = \text{width}(\mathcal{F}(I)) \leq \text{width}(\mathcal{C}(D_0)) \leq \text{width}(\mathcal{C}(D)) \leq |\mathcal{C}(D)| \leq 2^{2^{|I|}}.$$ 

\(\mathcal{C}(D)\) has chains of length \(\max(c, \text{rank}(D^+))\).
Proposition

Let $D$ be an infinite discrete group of finite rank. Then $G$ is isomorphic to

$$H \times \prod_{i=1}^{k} \mathbb{Z}(p_i^{\infty}) \times F$$

for not necessarily distinct primes $p_i$ and a subgroup $H$ of $\mathbb{Q}^m$ ($k, m \in \mathbb{N}_0$) and a finite group $F$. 
Proposition

Let $D$ be an infinite discrete group of finite rank. Then $G$ is isomorphic to

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for not necessarily distinct primes $p_i$ and a subgroup $H$ of $\mathbb{Q}^m$ ($k, m \in \mathbb{N}_0$) and a finite group $F$.

So $D$ contains an open subgroup isomorphic to $\mathbb{Z}$ or $\mathbb{Z}(p^\infty)$. 
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Recall that $\mathbb{Z}^\wedge \to \mathbb{T}$, $\chi \mapsto \chi(1)$ is a topological isomorphism.
Recall that $\mathbb{Z}^\wedge \rightarrow \mathbb{T}$, $\chi \mapsto \chi(1)$ is a topological isomorphism. The discrete topology $\delta$ on $\mathbb{Z}$ is the top element in $\mathcal{C}(\mathbb{Z})$ and $\sigma(\mathbb{Z}, \mathbb{T})$ is the bottom element.
Notation

Recall that $\mathbb{Z}^\wedge \to \mathbb{T}$, $\chi \mapsto \chi(1)$ is a topological isomorphism. The discrete topology $\delta$ on $\mathbb{Z}$ ist the top element in $\mathcal{C}(\mathbb{Z})$ and $\sigma(\mathbb{Z}, \mathbb{T})$ is the bottom element. $\sigma(\mathbb{Z}, \mathbb{T})$ is the topology of uniform convergence on the finite subsets of $\mathbb{T}$. A neighborhood basis is given by the sets $(F^a)_F$ where $F$ runs through all finite subsets of $\mathbb{T}$. 
Recall that $\mathbb{Z}^\wedge \rightarrow \mathbb{T}$, $\chi \mapsto \chi(1)$ is a topological isomorphism. The discrete topology $\delta$ on $\mathbb{Z}$ ist the top element in $C(\mathbb{Z})$ and $\sigma(\mathbb{Z}, \mathbb{T})$ is the bottom element. $\sigma(\mathbb{Z}, \mathbb{T})$ is the topology of uniform convergence on the finite subsets of $\mathbb{T}$. A neighborhood basis is given by the sets $(F^\triangleleft)_F$ where $F$ runs through all finite subsets of $\mathbb{T}$. So in order to find a lqc group topology strictly coarser than $\delta$ and strictly finer than $\sigma(\mathbb{Z}, \mathbb{T})$, we can try to take the topology of uniform convergence on a suitable (compact) subset between a finite set of $\mathbb{T}$ and $\mathbb{T}$. 
Recall that $\mathbb{Z}^\wedge \to \mathbb{T}$, $\chi \mapsto \chi(1)$ is a topological isomorphism. The discrete topology $\delta$ on $\mathbb{Z}$ is the top element in $C(\mathbb{Z})$ and $\sigma(\mathbb{Z}, \mathbb{T})$ is the bottom element. $\sigma(\mathbb{Z}, \mathbb{T})$ is the topology of uniform convergence on the finite subsets of $\mathbb{T}$. A neighborhood basis is given by the sets $F^a_F$ where $F$ runs through all finite subsets of $\mathbb{T}$.

So in order to find a lqc group topology strictly coarser than $\delta$ and strictly finer than $\sigma(\mathbb{Z}, \mathbb{T})$, we can try to take the topology of uniform convergence on a suitable (compact) subset between a finite set of $\mathbb{T}$ and $\mathbb{T}$.

Let us try to take a subsequence of $\left(\frac{1}{n} + \mathbb{Z}\right)_n$. 
Notation

Let \((b_n)_{n \in \mathbb{N}_0}\) be a strictly increasing sequence of natural numbers satisfying \(b_n \mid b_{n+1}\) and \(b_0 = 1\). We call such a sequence a \(D\text{-sequence}\).
Notation
Let \((b_n)_{n \in \mathbb{N}_0}\) be a strictly increasing sequence of natural numbers satisfying \(b_n \mid b_{n+1}\) and \(b_0 = 1\). We call such a sequence a \textit{D–sequence}.

Let

\[
S = \left\{ \frac{1}{b_n} + \mathbb{Z} : n \in \mathbb{N} \right\}
\]

and denote by \(\gamma_S\) the topology of uniform convergence on \(S\).
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Let \((b_n)_{n \in \mathbb{N}_0}\) be a strictly increasing sequence of natural numbers satisfying \(b_n | b_{n+1}\) and \(b_0 = 1\). We call such a sequence a \(D\text{-sequence}.\)

Let

\[ S = \left\{ \frac{1}{b_n} + \mathbb{Z} : n \in \mathbb{N} \right\} \]

and denote by \(\gamma_S\) the topology of uniform convergence on \(S\). A neighborhood basis at 0 in \(\gamma_S\) is given by the sequence of neighborhoods \((V_{S,k})_{k \in \mathbb{N}}\) where

\[ V_{S,k} = \left\{ j \in \mathbb{Z} : \frac{j}{b_n} + \mathbb{Z} \in T_k \right\} \]

and \(T_k = \{ x + \mathbb{Z} : |x| \leq \frac{1}{4k} \} \subseteq \mathbb{T}.\)
The Main Theorem

Theorem (Dikranjan, L.A.)

Let \((b_n)\) be a D–sequence. Denote by 
\[ S := \left\{ \frac{1}{b_n} + \mathbb{Z} : \ n \in \mathbb{N} \right\} \cup \{0\}. \]
The following conditions are equivalent:

(a) \(\gamma_S < \delta\);
(b) \(\left\{ \frac{b_{n+1}}{b_n} : \ n \in \mathbb{N} \right\}\) is unbounded.
(c) \(\sigma(\mathbb{Z}, \mathbb{T}) < \gamma_S \vee \sigma(\mathbb{Z}, \mathbb{T}) < \delta\).
(c) \implies (a) is trivial
(c) $\implies$ (a) is trivial
(a) $\implies$ (b) $\neg$(b) $\implies$ $\neg$(a)): Assume that $(\frac{b_{n+1}}{b_n})$ is bounded, i.e. there exists $m \in \mathbb{N}$ such that $\frac{b_{n+1}}{b_n} \leq m$ for all $n \in \mathbb{N}$. Fix $k \in V_{S,m}$. Assume that $k \neq 0$. Wlog $k > 0$. There exists a unique $n \in \mathbb{N}_0$ such that $\frac{b_n}{4} < k \leq \frac{b_{n+1}}{4}$ which is equivalent to $\frac{1}{4m} \leq \frac{b_n}{4b_{n+1}} < \frac{k}{b_{n+1}} \leq \frac{1}{4}$. Hence $\frac{k}{b_{n+1}} + \mathbb{Z} \not\in V_{S,m}$, a contradiction. This shows $V_{S,m} = \{0\}$. 

Compatible group topologies and the Mackey topology
(c) $\iff$ (a) is trivial
(a) $\iff$ (b) (¬(b) $\implies$ ¬(a)): Assume that $(\frac{b_{n+1}}{b_n})$ is bounded, i.e. there exists $m \in \mathbb{N}$ such that $\frac{b_{n+1}}{b_n} \leq m$ for all $n \in \mathbb{N}$. Fix $k \in V_{S,m}$.
Assume that $k \neq 0$. Wlog $k > 0$. There exists a unique $n \in \mathbb{N}_0$ such that $\frac{b_n}{4} < k \leq \frac{b_{n+1}}{4}$ which is equivalent to $\frac{1}{4m} \leq \frac{b_n}{4b_{n+1}} < \frac{k}{b_{n+1}} \leq \frac{1}{4}$. Hence $\frac{k}{b_{n+1}} + \mathbb{Z} \notin V_{S,m}$, a contradiction. This shows $V_{S,m} = \{0\}$.

(b) $\implies$ (c) Assume now that $(\frac{b_{n+1}}{b_n})$ is unbounded. Since the polar of every neighborhood of 0 in $\gamma_S \vee \sigma(\mathbb{Z}, \mathbb{T})$ is infinite, this topology is not precompact.
The crucial step is to prove that $\gamma_S \lor \sigma(\mathbb{Z}, \mathbb{T})$ is not discrete.

**Claim.** For every $y + \mathbb{Z} \in \mathbb{T}$ and an infinite subset $M_0 \subseteq M$ exists an infinite subset $M_1 \subseteq M_0$ such that for all $m, m' \in M_1$ we have $b_m - b_{m'} \in \{y + \mathbb{Z}\}^{\triangleleft}$. 
Non–large sets

Example
Let $b \geq 2$ be an integer, $(a_n)$ an increasing sequence in $\mathbb{N}$, and $b_n := b^{a_n}$. Then $(b_n)$ satisfies the hypothesis of the Theorem iff $(a_{n+1} - a_n)$ is unbounded.
Non–large sets

Example
Let \( b \geq 2 \) be an integer, \( (a_n) \) an increasing sequence in \( \mathbb{N} \), and \( b_n := b^{a_n} \). Then \( (b_n) \) satisfies the hypothesis of the Theorem iff \( (a_{n+1} - a_n) \) is unbounded.

Definition
A faithfully increasingly enumerated infinite subset \( A = \{ a_n : n \in \mathbb{N} \} \subseteq \mathbb{N} \) is called non-large if the differences \( (a_{n+1} - a_n) \) are unbounded. We put

\[
\mathcal{X} = \{ A \subseteq \mathbb{N} : A \text{ is infinite and non–large} \}.
\]
Non–large sets

Notation
Let \( p \) be a \((prime)\) number.
For \( A \in \mathcal{X} \) we define

\[
S_A = \left\{ \frac{1}{p^a} + \mathbb{Z} : a \in A \right\} \subseteq \mathbb{T}
\]

We denote the topology on \( \mathbb{Z} \) of uniform convergence on \( S_A \) by \( \gamma_A \) and by \( \sigma \) the topology \( \sigma(\mathbb{Z}, \mathbb{T}) \).
Comparing topologies

Notation
For \( A, B \in \mathcal{X} \) let

\[
\eta(B, A) := \sup \{ \min \{ a - b : a \in A, \ a \geq b \} : b \in B \} \in [0, \infty]
\]

Note that \( \eta : \mathcal{X} \times \mathcal{X} \to \mathbb{N} \cup \{\infty\} \) is not a symmetric function, yet it defines a sort of quasimetric on \( \mathcal{X} \) (with value \( \infty \) allowed).
Comparing topologies

Notation
For \( A, B \in \mathcal{X} \) let

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Note that \( \eta : \mathcal{X} \times \mathcal{X} \to \mathbb{N} \cup \{ \infty \} \) is not a symmetric function, yet it defines a sort of quasimetric on \( \mathcal{X} \) (with value \( \infty \) allowed).

Theorem
For \( A, B \in \mathcal{X} \) the following assertions are equivalent:

(a) \( \eta(B, A) < \infty \)

(b) \( \gamma_B \vee \sigma \leq \gamma_A \vee \sigma \)
Corollary

For $A, B \in \mathcal{X}$ the following assertions are equivalent:

(a) $\gamma_B \vee \sigma < \gamma_A \vee \sigma$.
(b) $\eta(B, A) < \eta(A, B) = \infty$.
Corollary

For \( A, B \in \mathbf{X} \) the following assertions are equivalent:

(a) \( \gamma_B \vee \sigma < \gamma_A \vee \sigma \).

(b) \( \eta(B, A) < \eta(A, B) = \infty \).

Lemma

There exists a sequence \((A_n)\) in \( \mathbf{X} \) such that \( \eta(A_n, A_{n+1}) < \infty \) and \( \eta(A_{n+1}, A_n) = \infty \) for all \( n \in \mathbb{N} \).
Chains

Corollary
For $A, B \in \mathcal{X}$ the following assertions are equivalent:

(a) $\gamma_B \vee \sigma < \gamma_A \vee \sigma$.
(b) $\eta(B, A) < \eta(A, B) = \infty$.

Lemma
There exists a sequence $(A_n)$ in $\mathcal{X}$ such that $\eta(A_n, A_{n+1}) < \infty$ and $\eta(A_{n+1}, A_n) = \infty$ for all $n \in \mathbb{N}$.

Corollary
There exists a poset embedding $(\mathbb{N}, \leq) \to (\mathcal{C}(\mathbb{Z}), \subseteq)$. In particular, $\mathbb{Z}$ contains chains of length $\omega$. 
Incomparable topologies

Corollary

For $A, B \in \mathcal{X}$ the following assertions are equivalent: $\gamma_A$ and $\gamma_B$ are incomparable if and only if $\eta(A, B) = \eta(B, A) = \infty$. 
Incomparable topologies

Corollary
For $A, B \in \mathcal{X}$ the following assertions are equivalent: $\gamma_A$ and $\gamma_B$ are incomparable if and only if $\eta(A, B) = \eta(B, A) = \infty$.

Lemma
There exists a family $(A_i)_{i \in I}$ in $\mathcal{X}$ such that for all $i \neq i'$ we have $\eta(A_i, A_{i'}) = \infty$ and $|I| = c$. 
Incomparable topologies

Corollary
For \( A, B \in \mathcal{X} \) the following assertions are equivalent: \( \gamma_A \) and \( \gamma_B \) are incomparable if and only if \( \eta(A, B) = \eta(B, A) = \infty \).

Lemma
There exists a family \((A_i)_{i \in I}\) in \( \mathcal{X} \) such that for all \( i \neq i' \) we have \( \eta(A_i, A_{i'}) = \infty \) and \( |I| = c \).

Corollary
\[ |\text{width}(\mathcal{C}(\mathbb{Z}))| \geq c. \]
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Theorem

Let $(b_n)$ be a D–sequence. Denote by $K := \{ \frac{1}{b_n} : n \in \mathbb{N} \} \cup \{0\}$.

The following conditions are equivalent:

(a) $\tau_K < \rho$;

(b) $\{ \frac{b_{n+1}}{b_n} : n \in \mathbb{N} \}$ is unbounded.

(c) $\sigma(\mathbb{R}, \mathbb{R}) < \tau_K \lor \sigma(\mathbb{R}, \mathbb{R}) < \rho$.

where $\rho$ denotes the usual topology on $\mathbb{R}$ and $\tau_K$ the topology of uniform convergence on $K$. 

Universität Passau
Comparable topologies on $\mathbb{Z}(p^\infty)$

**Theorem**

Let $(a_n)$ be a strictly increasing sequence of natural numbers and $C = \{p^{a_n} : n \in \mathbb{N}\} \subset \mathbb{Z} \subseteq \mathbb{Z}_p$. Let $\sigma = \sigma(\mathbb{Z}(p^\infty), \mathbb{Z}_p)$ denote the Bohr topology. The following assertions are equivalent:

(a) $\zeta_C < \delta$.

(b) The sequence $(a_{n+1} - a_n)$ is unbounded.

(c) $\sigma < \zeta_C \lor \sigma < \delta$

where $\delta$ denotes the discrete topology and $\sigma = \sigma(\mathbb{Z}(p^\infty), \mathbb{Z}_p)$ and $\zeta_C$ is the topology of uniform convergence on $C$. 
Let $G = \mathbb{R}^n \times H$ be a non-compact LCA group. Let $K$ be a compact open subgroup of $H$ and denote by $D = H/K$. Assume that $D$ is infinite or trivial. Let $\tilde{G} = \mathbb{R}^n \times D$; $C(G) \cong C(\tilde{G})$. 

1. Case: $\text{rank}(D) = \infty$. Then $\text{w}(\tilde{G}) = |D|$ and $2 |D| \leq \text{width} C(D) \leq \text{width} C(\tilde{G}) \leq |C(\tilde{G})| \leq 2^2 |D| = 2^2 |\text{w}(\tilde{G})|$. 

2. Case: $\text{rank}(D) < \infty$, $D$ is infinite. Then $\text{w}(\tilde{G}) = \omega$ and $D$ has an open subgroup of the form $D_1 \in \{\mathbb{Z}, \mathbb{Z}(p^\infty)\}$. 

$C(G) \cong C(\tilde{G})$. 

Compatible group topologies and the Mackey topology
Let $G = \mathbb{R}^n \times H$ be a non-compact LCA group. Let $K$ be a compact open subgroup of $H$ and denote by $D = H/K$. Assume that $D$ is infinite or trivial. Let $\tilde{G} = \mathbb{R}^n \times D$; $C(G) \cong C(\tilde{G})$

1. Case: $\text{rank}(D) = \infty$. Then $w(\tilde{G}) = |D|$ and

$$2^{2|D|} = \text{width} C(D) \leq \text{width} C(\tilde{G}) \leq |C(\tilde{G})| \leq 2^{w(\tilde{G})} = 2^{2|D|}$$
Let $G = \mathbb{R}^n \times H$ be a non-compact LCA group. Let $K$ be a compact open subgroup of $H$ and denote by $D = H/K$. Assume that $D$ is infinite or trivial. Let $\tilde{G} = \mathbb{R}^n \times D$; $C(G) \cong C(\tilde{G})$

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2. Case: $\text{rank}(D) < \infty$, $D$ is infinite. Then $w(\tilde{G}) = \omega$ and $D$ has an open subgroup of the form $D_1 \in \{\mathbb{Z}, \mathbb{Z}(p^\infty)\}$

$$c \leq \text{width}C(D_1) \leq \text{width}C(D) \leq \text{width}C(\tilde{G}) \leq |C(\tilde{G})| \leq 2^c$$
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$$c \leq \text{width} C(D_1) \leq \text{width} C(D) \leq \text{width} C(\tilde{G}) \leq |C(\tilde{G})| \leq 2^c$$
3. Case: $D = \{0\}$ and $n \geq 1$. Then $w(G) = \omega$ and $\mathbb{R}$ is a factor of $G$.

$$c \leq \text{width } \mathcal{C}(\mathbb{R}) \leq \text{width } \mathcal{C}(G) \leq |\mathcal{C}(G)| \leq 2^c$$
Let \((V, \tau)\) be a topological vector space. Under which conditions exists a finest (locally convex) vector space topology \(\mu\) on \(V\) such that the topological duals \((V, \tau)'\) and \((V, \mu)'\) coincide? There are three important results:
Let \((V, \tau)\) be a topological vector space. Under which conditions exists a finest (locally convex) vector space topology \(\mu\) on \(V\) such that the topological duals \((V, \tau)'\) and \((V, \mu)'\) coincide? There are three important results:

1. **Mackey, 1946** In the class of all locally convex vector space topologies there always exists a finest locally convex vector space topology, it is called the *Mackey topology*.

2. **Kakol, 1987** In the class of all topological vector spaces such a finest vector space topology does not exist in general.

3. If \((V, \tau)\) is a metrizable locally convex vector space then \(\tau\) is the Mackey topology.
In 1964, Varopoulos studied the counterpart of (1) and (3) for the class of all locally precompact abelian groups and proved that any metrizable locally precompact group $G$ is the finest locally precompact group topology having the character group $G^\wedge$. 
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Chasco, Martín Peinador and Tarieladze started in 1999 to study the question whether an analogue of the Mackey topology exists on topological group. Motivated by (2), they restricted themselves to locally quasi–convex groups.
Proof of the existence of the Mackey topology

Let \( (\tau_i)_{i \in I} \) be the family of all locally convex vector space topologies on \( V \) such that \( (V, \tau)' = (V, \tau_i)' \) and consider

\[
V \longrightarrow \prod_{i \in I} (V, \tau_i), \quad v \mapsto (v)_{i \in I}.
\]

The topology \( \mu \) on \( V \) induced by this embedding is the supremum of all the \( \tau_i \). As a consequence of the Hahn–Banach theorem, \( (V, \mu)' = (V, \tau)' \). Obviously, \( \mu \) is the finest locally convex topology with this property.
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Definition
A subgroup $H$ of a topological group $G$ is called **dually embedded** if every continuous character of $H$ can be extended to a continuous character of $G$. 
Example

The subgroup $H = \mathbb{Z}^{(\mathbb{N})}$ of the Hilbert space $\ell^2$ is not dually embedded.

Proof.

$H$ is discrete, so every sequence $(z_n) \in \mathbb{T}^\mathbb{N}$ gives rise to a continuous character: $H \to \mathbb{T}$, $\sum_n k_n e_n \mapsto \sum k_n z_n + \mathbb{Z}$. In particular, the sequence $(\frac{1}{2} + \mathbb{Z})_n$ gives rise to the character $\sum_n k_n e_n \mapsto \frac{1}{2} k_n + \mathbb{Z}$.

The characters of $\ell^2$ are of the form $\ell^2 \to \mathbb{T}$, $x \mapsto f(x) + \mathbb{Z}$ where $f \in (\ell^2)' \cong \ell^2$. Since $(\frac{1}{2}) \notin \ell^2$, the above character of $H$ cannot be extended to a character of $\ell^2$. \qed
Problem (Mackey topology problem)

Does every lqc group admit a Mackey topology? This is equivalent to: Is the supremum of any two compatible group topologies again compatible?
Remark

Let $(E, \tau)$ be a locally convex vector space. Let $\eta$ be compatible with $\tau$. Then for every neighborhood $U$ of $0$ in $(E, \eta)$ the polar $U^\circ = \{ \phi \in E' : \forall x \in U \ |\phi(x)| \leq 1 \}$ is an absolutely convex set. Further, by the Banach–Alaoglu theorem, the polar $U^\circ$ is a compact subset of $(E', \sigma(E', E))$. 
Remark
Let $(E, \tau)$ be a locally convex vector space. Let $\eta$ be compatible with $\tau$. Then for every neighborhood $U$ of 0 in $(E, \eta)$ the polar $U^\circ = \{ \varphi \in E' : \forall x \in U \ |\varphi(x)| \leq 1 \}$ is an absolutely convex set. Further, by the Banach–Alaoglu theorem, the polar $U^\circ$ is a compact subset of $(E', \sigma(E', E))$.

Theorem (Mackey, Arens)
The Mackey topology of a locally convex vector space $(E, \tau)$ can be described explicitly, it is the topology $\tau_{ac}$ of uniform convergence on all absolutely convex $(E', \sigma(E', E))$–compact subsets of $E'$. 
Proof.
We have already seen that any locally convex vector space topology on $E$ compatible with $\tau$ is coarser than $\tau_{ac}$. In order to prove that the set of compatible topologies has a top element it is therefore sufficient to verify that $\tau_{ac}$ is compatible. Once this is shown, $\tau_{ac}$ is the finest compatible topology.
So fix a absolutely convex $\sigma(E', E)$–compact set $S$. It suffices to show that every linear form $f : E \to K$ for which $f(S^\circ)$ is bounded, belongs to $E'$. In order to do so, we consider two dual pairs: $(E, E')$ and $(E, E^*)$. We denote the polars formed in the dual pair $(E, E')$ by $^\circ$ and the those formed in $(E, E^*)$ by $^\bullet$. First of all, for a subset $A \subseteq E'$, we have $A^\circ = A^\bullet$, since the polar is formed in $E$. We can consider $(E', \sigma(E', E))$ as a subspace of $(E^*, \sigma(E^*, E))$. So $S$ is also compact and absolutely convex when considered as a subset of $E^*$. We obtain $S \subseteq S^{^\circ\circ} = S^{^\bullet\bullet} = S^{^\bullet\bullet} \cap E' \subseteq S^{^\bullet\bullet} = S$.
The last equation holds by the bipolar theorem. So we see that $\tau_{ac}$ is compatible.
Definition (Chasco, Martín Peinador, Tarieladze; 1999)
A lqc group $G$ is called $g$–barrelled if every $\sigma(G^\wedge, G)$ compact subset is equicontinuous.

Proposition (Chasco, Martín Peinador, Tarieladze; 1999)
If $(G, \tau)$ is $g$–barrelled, then $\tau$ is the Mackey topology.
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Proposition (Chasco, Martín Peinador, Tarieladze; 1999)
If $(G, \tau)$ is $g$–barrelled, then $\tau$ is the Mackey topology.

Examples (Chasco, Martín Peinador, Tarieladze; 1999)
The following groups are $g$–barrelled:
1. Lqc completely metrizable groups,
2. Lqc Cech complete groups,
3. LCA groups, and
4. (locally) pseudocompact groups.
Example (Bonales Trigos, Mendoza; 2003)

1. $(G, \tau)$ is the Mackey topology.
2. There exists a $\sigma(G^\wedge, G)$–compact quasi–convex subset $C \subseteq G^\wedge$ which is not equicontinuous; hence $G$ is not $g$–barrelled.
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Motivation

Theorem

1. Every metrizable locally convex vector space is Mackey (in the class of all locally convex vector spaces).
2. Every completely metrizable lqc group is Mackey (in the class of lqc groups).
3. Varopoulos Every metrizable locally precompact abelian group is the Mackey topology in the class of all locally precompact abelian groups.
Motivation

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Question

Is every metrizable lqc group Mackey (in the class of lqc groups)?
Motivation

Theorem

1. Every metrizable locally convex vector space is Mackey (in the class of all locally convex vector spaces).
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3. Varopoulos Every metrizable locally precompact abelian group is the Mackey topology in the class of all locally precompact abelian groups.

Question

Is every metrizable lqc group Mackey (in the class of lqc groups)?
In particular, is every precompact group \((G, \sigma)\) with countable dual group \(G^\wedge\) Mackey?
Counterexamples

Example (Martín Peinador)
The group $c_0(\mathbb{T})$ with the weak group topology is precompact and metrizable but not Mackey, since the group $c_0(\mathbb{T}) = c_0(\mathbb{R})/\mathbb{Z}^{(\mathbb{N})}$ with the quotient topology is complete and metrizable and has the same dual group.
Counterexamples

Example (Martín Peinador)
The group $c_0(\mathbb{T})$ with the weak group topology is precompact and metrizable but not Mackey, since the group $c_0(\mathbb{T}) = c_0(\mathbb{R})/\mathbb{Z}^{(\mathbb{N})}$ with the quotient topology is complete and metrizable and has the same dual group.

Theorem (Dikranjan, Martín Peinador, Tarieladze, 2010)
Let $X$ be an infinite abelian compact connected metrizable group. Denote by $c_0(X)$ the group of all null–sequences in $X$. $c_0(X)$ endowed with the topology of uniform convergence is a complete and metrizable lqc (hence Mackey) group. The dual of $c_0(X)$ is countable. Hence the weak group topology on $c_0(X)$ is a strictly coarser metrizable group topology (with the same dual group) and hence the weak topology is not Mackey.
A group topology on an abelian group is called **linear** if it has a neighborhood basis consisting of open subgroups.
-notch

A group topology on an abelian group is called **linear** if it has a neighborhood basis consisting of open subgroups.

**Theorem (de la Barrera Mayoral, L.A., 2011)**

Let $\lambda$ be a non–discrete linear group topology on $\mathbb{Z}$. Then $(\mathbb{Z}, \lambda)$ is a precompact, metrizable group topology which is not Mackey.
Question
Let $\lambda_2$ denote the 2-adic topology on $\mathbb{Z}$.
For every sequence $(2^{a_n})$ where $(a_n)$ is strictly increasing sequence of natural numbers and $a_{n+\ell} - a_n \to \infty$ for some $\ell \in \mathbb{N}$, the topology of uniform convergence on $\{2^{-a_n} + \mathbb{Z} : n \in \mathbb{N}\}$ is compatible with $\lambda_2$. Is the supremum of all these topologies the Mackey topology?
Theorem (de la Barrera Mayoral, 2014)

*The torsion subgroup of $\mathbb{T}$ is not Mackey.*
Theorem (de la Barrera Mayoral, 2014)

The torsion subgroup of $\mathbb{T}$ is not Mackey.
The group $\mathbb{Q}$ of rational numbers with the topology induced by $\mathbb{R}$ is not Mackey.
Varopoulos proved that a metrizable locally precompact group topology is the Mackey topology within the class of all locally precompact group topologies.
Varopoulos proved that a metrizable locally precompact group topology is the Mackey topology within the class of all locally precompact group topologies.

**Question**

Describe all abelian groups such that every metrizable lqc group topology is Mackey.
Lemma

Every lqc group topology on a bounded group is linear.
Lemma

Every lqc group topology on a bounded group is linear.

Proof.
Suppose that $G$ is a bounded group of exponent $m$, i.e. $mx = 0$ for all $x \in G$. Let $\tau$ be a lqc group topology on $G$. Fix a qc neighborhood $U$ of 0 in $(G, \tau)$. Let $W$ be a qc neighborhood which satisfies $W + \ldots + W \subseteq U$. For $x \in W$ we have $\langle x \rangle \subseteq U$ and hence $\chi(x) = 0$ for all $\chi \in U^\triangleright$. Hence $W \subseteq \bigcap_{\chi \in U^\triangleright} \ker(\chi) \subseteq U$. So the latter set is an open subgroup. \qed
Bounded metrizable groups

Proposition

Every bounded metrizable lqc group is Mackey.

Proof.

Assume that $(G, \tau)$ is a bounded lqc metrizable group. Let $\tau'$ be a compatible topology. By the above Lemma $\tau$ and $\tau'$ are linear.

Let $H \leq G$ be a $\tau'$–open subgroup. It is sufficient to show that $H$ is $\tau$–open. Since every homomorphism $\chi: G \to T$ which is trivial on $H$ is $\tau'$–continuous and hence $\tau$–continuous, $H$ is in particular $\tau$–closed. Assume that $H$ is not $\tau$–open. Hence $G/H$ with the quotient topology $\mathbb{Q}(\tau)$ induced by $\tau$ is a non–discrete metrizable group such that every homomorphism into $T$ is continuous. This means that the Bohr–topology is coarser than $\mathbb{Q}(\tau)$. But this is a contradiction. A non-discrete metrizable group has many non-trivial convergent sequences, the Bohr topology has only those which are eventually constant. Contradiction. So $H$ is $\tau$–open.
Bounded metrizable groups

Proposition

Every bounded metrizable lqc group is Mackey.

Proof.

Assume that \((G, \tau)\) is a bounded lqc metrizable group. Let \(\tau'\) be a compatible topology. By the above Lemma \(\tau\) and \(\tau'\) are linear. Let \(H \leq G\) be a \(\tau'\)–open subgroup. It is sufficient to show that \(H\) is \(\tau\)–open. Since every homomorphism \(\chi : G \to \mathbb{T}\) which is trivial on \(H\) is \(\tau'\)–continuous and hence \(\tau\)–continuous, \(H\) is in particular \(\tau\)–closed. Assume that \(H\) is not \(\tau\)–open. Hence \(G/H\) with the quotient topology \(Q(\tau)\) induced by \(\tau\) is a non–discrete metrizable group such that every homomorphism into \(\mathbb{T}\) is continuous. This means that the Bohr–topology is coarser than \(Q(\tau)\). But this is a contradiction. A non-discrete metrizable group has many non-trivial convergent sequences, the Bohr topology has only those which are eventually constant. Contradiction. So \(H\) is \(\tau\)–open.
Theorem (L.A., de la Barrera Mayoral, Dikranjan, Martín Peinador, 2016)

An abelian group has the property that every metrizable lqc group topology is Mackey iff $G$ is bounded.

Proof. Let $G$ be an unbounded group. Then $G$ has a subgroup of the form
1. $\mathbb{Z}$
2. $\mathbb{Z}(p^{\infty})$ or
3. $\bigoplus_{n \in \mathbb{N}} \mathbb{Z}/m_n \mathbb{Z}$ where $(m_n)_{n \in \mathbb{N}}$ is a strictly increasing sequence.

Each of these groups admits a metrizable group topology which is not Mackey.
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1. \( \mathbb{Z} \) or
2. \( \mathbb{Z}(p^{\infty}) \) or
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Each of these groups admits a metrizable group topology which is not Mackey. \( \square \)
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Motivation

Consider the group $\mathbb{R}^{(\mathbb{N})}$ of all finite real sequences with the topology induced by the product topology. Then

1. $\mathbb{R}^{(\mathbb{N})}$ is a metrizable locally convex vector space and hence a Mackey space.

2. Gabriyelyan $\mathbb{R}^{(\mathbb{N})}$ is not Mackey group.
Motivation

Consider the group $\mathbb{R}^\mathbb{N}$ of all finite real sequences with the topology induced by the product topology. Then

1. $\mathbb{R}^\mathbb{N}$ is a metrizable locally convex vector space and hence a Mackey space.
2. Gabriyelyan $\mathbb{R}^\mathbb{N}$ is not Mackey group.

Mackeyness depends on the class of groups which is considered.
Barr and Kleisli’s categorical approach

Let $\mathcal{I}$ be a class of lqc groups which is stable under

1. taking products,
2. taking subgroups and which satisfies
3. $T \in \mathcal{I}$. 

Examples

For $\mathcal{I}$ one can take the class of all

1. lqc groups,
2. precompact groups,
3. all nuclear groups (they form a Hausdorff variety which
   contains all LCA groups and all locally convex nuclear
   vector spaces).
Barr and Kleisli’s categorical approach

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Examples

For $\mathcal{I}$ one can take the class of all

1. lqc groups,
2. precompact groups,
3. all nuclear groups (they form a Hausdorff variety which contains all LCA groups and all locally convex nuclear vector spaces).
Definition

Let \((G, \tau) \in \mathcal{I}\). A topology \(\mu\) on \(G\) such that \((G, \mu) \in \mathcal{I}\) is called \(\mathcal{I}-\text{Mackey group for}\) \((G, \tau)\) if it is the finest group topology in \(\mathcal{I}\) which is compatible with \(\tau\).
**Definition**
Let \((G, \tau) \in \mathcal{S}\). A topology \(\mu\) on \(G\) such that \((G, \mu) \in \mathcal{S}\) is called \(\mathcal{S}\)–Mackey group for \((G, \tau)\) if it is the finest group topology in \(\mathcal{S}\) which is compatible with \(\tau\).

**Definition**
A subgroup \(H\) of a topological group \(G\) is called **dually embedded** if every continuous character of \(H\) can be extended to a continuous character of \(G\).
Definition
Let \((G, \tau) \in \mathcal{I}\). A topology \(\mu\) on \(G\) such that \((G, \mu) \in \mathcal{I}\) is called \(\mathcal{I}\)–Mackey group for \((G, \tau)\) if it is the finest group topology in \(\mathcal{I}\) which is compatible with \(\tau\).

Definition
A subgroup \(H\) of a topological group \(G\) is called **dually embedded** if every continuous character of \(H\) can be extended to a continuous character of \(G\).

Notation
We say a class \(\mathcal{I}\) as above has the (DE) property if for every group \((G, \tau) \in \mathcal{I}\) every subgroup \(H\) of \(G\) is dually embedded.
Theorem (Barr, Kleisli; 2001)

Let $\mathcal{S}$ be a class as above and suppose that it has the (DE) property. Then the following holds:

For every $(G, \tau) \in \mathcal{S}$ exists a $\mathcal{S}$–Mackey group topology; it will be denoted by $\mu(\tau)$.

If $(G_1, \tau_1)$ and $(G_2, \tau_2)$ are in $\mathcal{S}$ and if $\varphi : (G_1, \tau_1) \to (G_2, \tau_2)$ is a continuous homomorphism, then also $\varphi^\mu : (G_1, \mu(\tau_1)) \to (G_2, \mu(\tau_2))$, $x \mapsto \varphi(x)$ is continuous.
Definition
If a class of groups $\mathcal{S}$ has the properties stated in the theorem, the application

$$\mathcal{S} \rightarrow \mathcal{S}, \ (G, \tau) \mapsto (G, \mu(\tau))$$

is called **Mackey coreflection**.

Theorem (Barr, Kleisli)
*If a class of groups $\mathcal{S}$ has a Mackey coreflection, then it has the property (DE).*

Example
The class of all nuclear groups has the property (DE), so within this class exists a Mackey coreflection.
The class of linearly topologized groups

Notation
A group topology on an abelian group is called **linear** if it has a neighborhood basis consisting of open subgroups. We denote by $\mathcal{L}_I N$ the class of all Hausdorff groups endowed with a linear topology.

Proposition
Every linear Hausdorff group topology is lqc.
The class of linearly topologized groups

Notation
A group topology on an abelian group is called \textit{linear} if it has a neighborhood basis consisting of open subgroups. We denote by \( \mathcal{LIN} \) the class of all Hausdorff groups endowed with a linear topology.

Proposition
Every linear Hausdorff group topology is lqc. The class \( \mathcal{LIN} \) has the property (DE).
Proof.
Let $\tau$ be a linear group topology on $G$ and let $H \leq G$ be a subgroup. Let $\chi : H \to \mathbb{T}$ be a continuous character. By assumption, there exists an open subgroup $U$ of $G$ such that $\chi(U \cap H) \subseteq \mathbb{T}^+$ and hence $\chi(U \cap H) = \{0\}$.
Since $H/H \cap U$ embeds in $G/U$ and both groups are divisible, the homomorphism $\tilde{\chi} : H/H \cap U \to \mathbb{T}$, $x + H \cap U \mapsto \chi(x)$ can be extended to a homomorphism $\tilde{\chi} : G/U \to \mathbb{T}$ (since the torus is divisible). So the composition of the projection $G \to G/U$ with $\tilde{\chi}$ has the desired properties.

\[\square\]
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Let $\tau$ be a linear group topology on $G$ and let $H \leq G$ be a subgroup. Let $\chi : H \rightarrow \mathbb{T}$ be a continuous character. By assumption, there exists an open subgroup $U$ of $G$ such that $\chi(U \cap H) \subseteq \mathbb{T}^+$ and hence $\chi(U \cap H) = \{0\}$.
Since $H/H \cap U$ embeds in $G/U$ and both groups are divisible, the homomorphism $\tilde{\chi} : H/H \cap U \rightarrow \mathbb{T}$, $x + H \cap U \mapsto \chi(x)$ can be extended to a homomorphism $\tilde{\chi} : G/U \rightarrow \mathbb{T}$ (since the torus is divisible). So the composition of the projection $G \rightarrow G/U$ with $\tilde{\chi}$ has the desired properties.

\[ \square \]

Corollary

The Mackey topology exists in $\mathcal{L} \mathcal{I} \mathcal{N}$.
**Definition**
Let \((G, \tau)\) be an abelian topological group. A subgroup \(H\) of \(G\) is called **\(B\)-embedded** if every homomorphism \(\chi : G \to \mathbb{T}\) which is trivial on \(H\) is continuous.

Remark 1. Every open subgroup is \(B\)-embedded.
Remark 2. Every \(B\)-embedded subgroup is dually closed.
Remark 3. The set of \(B\)-embedded subgroups of a group \((G, \tau)\) is closed under taking finite intersections. So the \(B\)-embedded subgroups form a neighborhood basis of a linear group topology \(\lambda_B\) on \(G\).
Remark 4. If \(\tau\) and \(\tau'\) are compatible (linear) topologies on \(G\) then \((G, \tau)\) and \((G, \tau')\) have the same \(B\)-embedded subgroups.
Definition
Let \((G, \tau)\) be an abelian topological group. A subgroup \(H\) of \(G\) is called \textbf{\textit{B–embedded}} if every homomorphism \(\chi : G \to \mathbb{T}\) which is trivial on \(H\) is continuous.

A subgroup \(H\) of \(G\) is called \textbf{\textit{dually closed}}, if for every \(x \in G \setminus H\) there exists a character \(\chi \in G^\wedge\) such that \(\chi(H) = \{0\}\) and \(\chi(x) \neq 0\).
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Remark

1. Every open subgroup is B–embedded.
2. Every B–embedded subgroup is dually closed.
3. The set of B–embedded subgroups of a group \((G, \tau)\) is closed under taking finite intersections. So the B–embedded subgroups form a neighborhood basis of a linear group topology \(\lambda^B_\tau\) on \(G\).
4. If \(\tau\) and \(\tau'\) are compatible (linear) topologies on \(G\) then \((G, \tau)\) and \((G, \tau')\) have the same B–embedded subgroups.
Theorem (L.A., Dikranjan, 2016)

A linear group \((G, \tau)\) is Mackey iff every \(B\)--embedded subgroup is open. Moreover, the mapping

\[
\mathcal{LIN} \longrightarrow \mathcal{LIN}, (G, \tau) \longmapsto (G, \lambda^B_{\tau})
\]

is a Mackey–coreflection.
Lemma

Every lqc group topology on a bounded group is linear.

Question

Let $m \geq 2$. Is $(\mathbb{Z}/m\mathbb{Z})^c$ (with the topology induced by the product topology) Mackey?
Lemma

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A bounded lqc group \((G, \tau)\) is Mackey in the class of all lqc groups iff every B–embedded subgroup is open.
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Let \(m \geq 2\). Is \((\mathbb{Z}/m\mathbb{Z})^{(c)}\) (with the topology induced by the product topology) Mackey?
Theorem

Let \((G, \tau)\) be a bounded lqc group such that \(|H^\wedge| < \mathfrak{c}\) for every countable subgroup \(H\) of \(G\). Then \(G\) is precompact and Mackey.

Proof.

\(\tau\) is a linear topology.

We shall show that every \(B\)–embedded subgroup has finite index: Let \(U\) be a \(B\)–embedded subgroup. Assume that \([G : U] = \infty\). Wlog we may assume that \([G : U] = \omega\). Choose a countable subgroup \(H\) of \(G\) such that \(G = H + U\). Then \(H/H \cap U \to G/U, \ x + H \cap U \mapsto x + U\) is a continuous isomorphism. Since \(U\) is \(B\)–embedded, every homomorphism \(G/U \to \mathbb{T}\) and hence also \(H/H \cap U \to \mathbb{T}\) is continuous. So

\[
\mathfrak{c} > |H^\wedge| \geq |(H/H \cap U)^\wedge| \geq |(G/U)^\wedge| \geq \mathfrak{c}
\]

This contradiction shows that \([G : U] < \infty\).

Every \(B\)–embedded subgroup is open and hence \(\tau\) is Mackey.
Corollary

Let \((G, \tau)\) be a bounded precompact group such that every countable subgroup of \(G\) is metrizable. Then \(G\) is Mackey.
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Corollary

\((\mathbb{Z}/m\mathbb{Z})^{(c)}\) is Mackey.
Theorem (Diaz Nieto, Martín Peinador, 2014)

Let $G$ be a lqc group and let $H$ be a closed subgroup such that $G/H$ is lqc. If $G$ is Mackey, then $G/H$ is Mackey as well.
A group which has no Mackey topology

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Let $G$ be an abelian Hausdorff group and denote by $A(G)$ the free abelian topological group over $G$; then $A(G) \rightarrow G$, $\sum k_g e_g \mapsto k_g g$ is a projection. $A(G)$ is lqc.
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Remark
If there exists a lqc group without Mackey topology, there exists a free abelian group without Mackey topology.
Theorem (Graev)

For a Tychonoff space $X$ with distinguished point $p \in X$ there exists (up to topological isomorphism) a unique topological group $(A(X; p), \tau)$ which is characterized by the following universal property:

- algebraically, $A(X)$ is a free abelian group;
- there is an embedding $\eta : X \to A(X)$ such that $\eta(p) = 0$ and $\eta(X \setminus \{p\})$ is a basis of $A(X; p)$;
- for every continuous mapping $f : X \to H$ with $f(p) = 0_H$ where $H$ is an abelian Hausdorff group with neutral element $0_H$, the unique homomorphism $f' : A(X; p) \to H$ which satisfies $f' \circ \eta = f$ is continuous.
Notation

Let $s$ be a convergent sequence, e.g. $s = \{0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$.

Corollary

$A(s;0)^{\wedge} \cong c_{0}(\mathbb{T})$. 
The Main Theorem

**Theorem**

\[ A(s;0) \text{ is not Mackey.} \]
The Main Theorem

Theorem

$A(s; 0)$ is not Mackey.

Equivalently,

Let $G = \mathbb{Z}^{(\mathbb{N})}$ be endowed with the topology $\sigma := \sigma(\mathbb{Z}^{(\mathbb{N})}, c_0(\mathbb{T}))$, where we identify $(x_n + \mathbb{Z}) \in c_0(\mathbb{T})$ with the character

$$\mathbb{Z}^{(\mathbb{N})} \to \mathbb{T}, \quad (k_n) \mapsto \sum k_n x_n + \mathbb{Z}$$

Then $(G, \sigma)$ is not Mackey.
Proof.
For $\alpha \in \mathbb{R} \setminus \{0\}$ denote by

$$S_\alpha = \{(x_n) \in c_0(T) : |\text{supp}(x_n)| \leq 1 \text{ and } x_n \in \{0, \alpha, -\alpha\}\}$$

and let $\tau_\alpha$ be the topology of uniform convergence on $S_\alpha$. If $\alpha \notin \mathbb{Q}$, then $\tau_\alpha \vee \sigma$ is compatible with $\sigma$. BUT $(\tau_\alpha \vee \sigma) \vee (\tau_{\alpha + \frac{1}{2}} \vee \sigma) = \tau_\alpha \vee \tau_{\alpha + \frac{1}{2}} \vee \sigma$ is not compatible. \qed
Thank you for your attention.