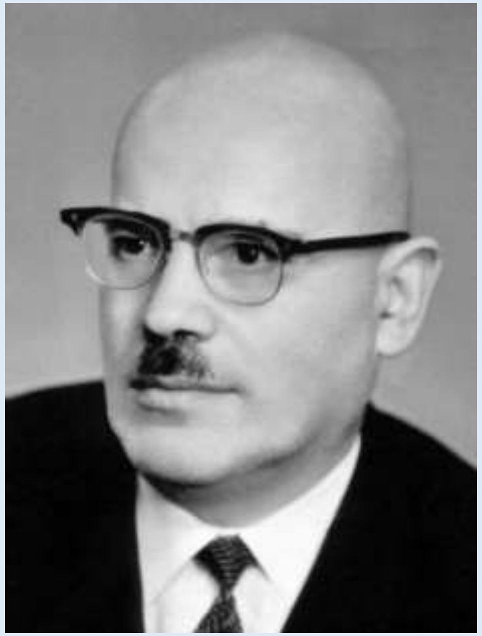
Banach Spaces and their Applications

International Conference dedicated to the 70^{th} birthday of Anatolij Plichko

26-29 June 2019
Lviv, Ukraine

A sketch of the History of **Abstract Functional Analysis in Ukraine:** from Banach to Plichko

Organizing Institutions

Ivan Franko National University of Lviv

Vasyl Stefanyk Precarpathian National University

Pidstryhach Institute for Applied Problems of Mechanics and Mathematics of National Academy of Sciences of Ukraine

Yurii Fedkovych Chernivtsi National University


Kyiv
Kharkiv
Lviv
Ivano-Frankivsk
Chernivtsi
Chervona
Kamyanka

In XIX century and till the end of the IWW, Lviv and Chernivtsi were important cities of the Austro-Hungarian Empire, being capitals of Galicia and Bukovina provinces

The Lviv University is the oldest university in Ukraine,
which traces its history from Jesuit Collegium founded in 1608.
In 1661 the Polish King Jan Kazimierz
granted the title of University to the Collegium.

The building of the former Jesuit convictus was given to the Lviv university in 1851.

In times of Banach, mathematicians worked in this building.

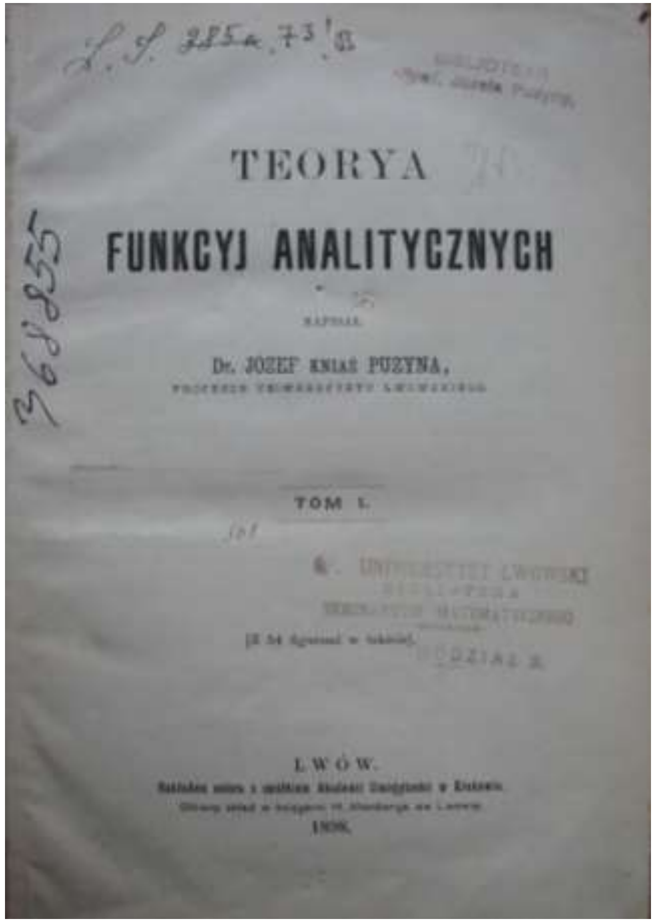
In 1920, the Lviv University moved to the former building of the Galician Parliament.

Many brilliant mathematicians worked at the Lviv University during its long history.
The most famous is of course Stefan Banach.

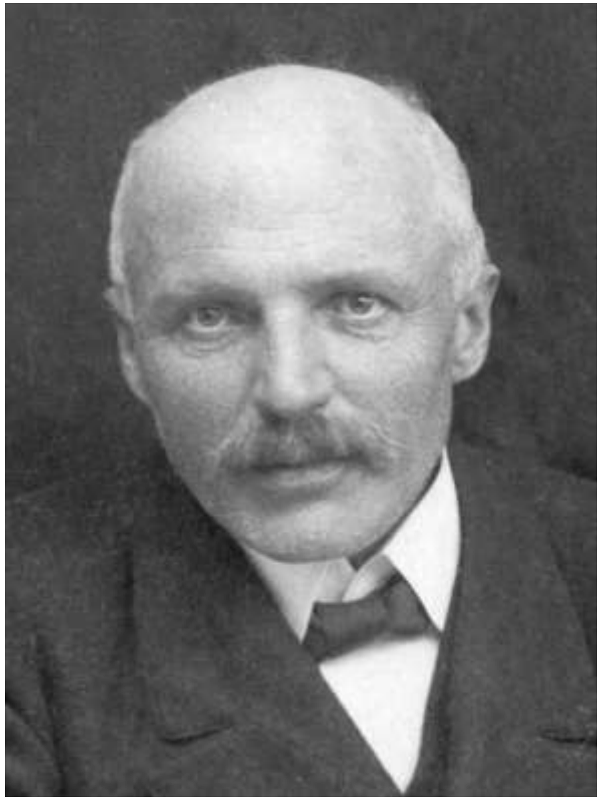
But the phenomenon of Banach and Lwów mathematical school could not appear without intensive preliminary work of mathematicians of Austrian time.

Stefan Banach (1892 – 1945)

A key role in development of Lviv mathematics in Austrian time belongs to **Józef Puzyna**, professor of mathematics,
brilliant lector and organizer of science,
Rector of the Lviv university in 1905.


TEORYA
FUNKCYJ ANALITYCZNYCH
Dr. JÓZEF KNIAŹ PUZYNA,
TOM I.
LWÓW.

K. Kuratowski wrote that J. Puzyna was a precursor, whose ideas were developed by the next generation of Polish mathematicians.


Jósef Puzyna (1856–1919)

The main J. Puzyna's work was two-volume monograph
„Theory of Analytic Functions" published in 1898.

At the same time in Bukovina
(another province of Austro-Hungarian Empire),
a key person in developing modern mathematics
was Professor **Hans Hahn**
who worked in the Chernivtsi University
during 1909 – 1915.

Hans Hahn is known due to:

- Hahn-Banach Theorem on extension of linear functionals;
- Uniform Boundedness Principle (proved by Hahn independently of Banach and Steinhaus);
- Hahn decomposition theorem for sign-measure;
- Hahn-Kolmogorov Theorem on extension of a measure from an algebra to a σ-algebra;
- Hahn-Mazurkiewicz Theorem on Peano continua;
- Vitali-Hahn-Saks Theorem on convergence of measures.

Hans Hahn (1879 – 1934)

In 1908 Józef Puzyna invited a young talented mathematician **Wacław Sierpiński** for a job in the Lviv University.
Sierpiński graduated from Warsaw University and was a pupil of a brilliant Ukrainian mathematician **Georgiy Voronoi** (who worked in Warsaw University during 1896 – 1905).

Georgiy Voronoi (1868 – 1908)

Wacław Sierpiński (1882 – 1969)

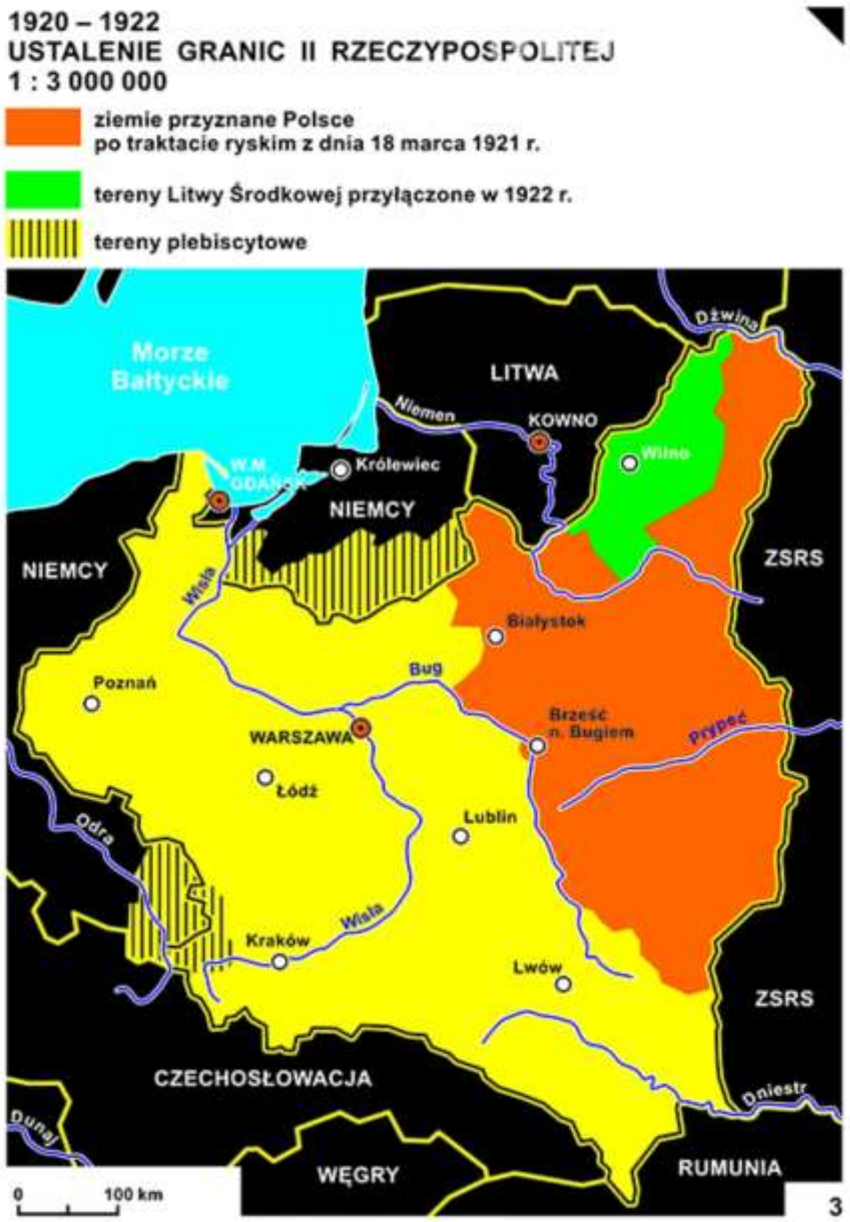
Waclaw Sierpiński worked at
the Lviv University during 1908 – 1914.

Together with Puzyna,
he guided the scientific seminars preparing
new generations of Lviv mathematicians.

Sierpiński taught many lecture courses
including the modern courses in just created
Set Theory and Theory of Lebesgue Measure.

This was that fertile ground on which the next generation
of Lviv mathematicians graciously rocketed creating what
is known as the Lwów mathematical school
of interwar period.

Wacław Sierpiński (1882 – 1969)



Then there was the IWW that dramatically changed the political map of Europe.

The Austro-Hungarian Empire disappeared,
the Bolshevik Revolution of 1917 turned
Russian Empire into communist USSR.

Attempts to create an independent Ukrainian country in Galicia and Bukovina and also in Central Ukraine failed (because of many reasons).

Lwów became one of the most important cities
of newly recreated Polish state,
whereas Bukovina (together with Chernivtsi)
was attached to Romania.

In spite of quite uncertain situation in politics,
in Mathematics everything was very optimistic.

Lwów Mathematical School

Hugo Steinhaus
(1887 – 1972)

Stefan Banach
(1992 – 1945)

Władysław Orlicz
(1903 – 1990)

Stanisław Mazur
(1905 – 1981)

Stanisław Ulam
(1909 – 1984)

Lwów Mathematical School

Hugo Steinhaus
(1887 – 1972)

Władysław Orlicz
(1903 – 1990)

Stanisław Mazur
(1905 – 1981)

Stanisław Ulam
(1908 – 1984)

Mark Kac
(1914 – 1984)

Stanisław Ruziewicz
(1889 – 1941)

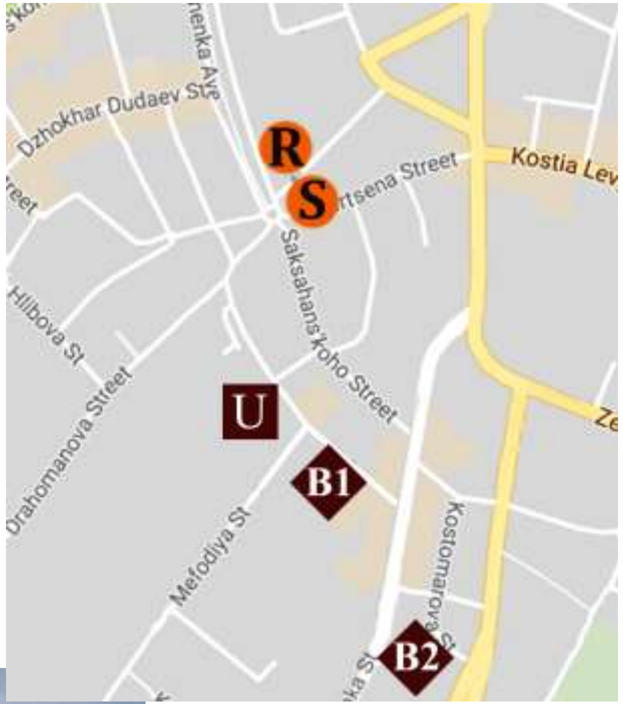
Herman Auerbach
(1901 – 1942)

Stanisław Saks
(1897 – 1942)

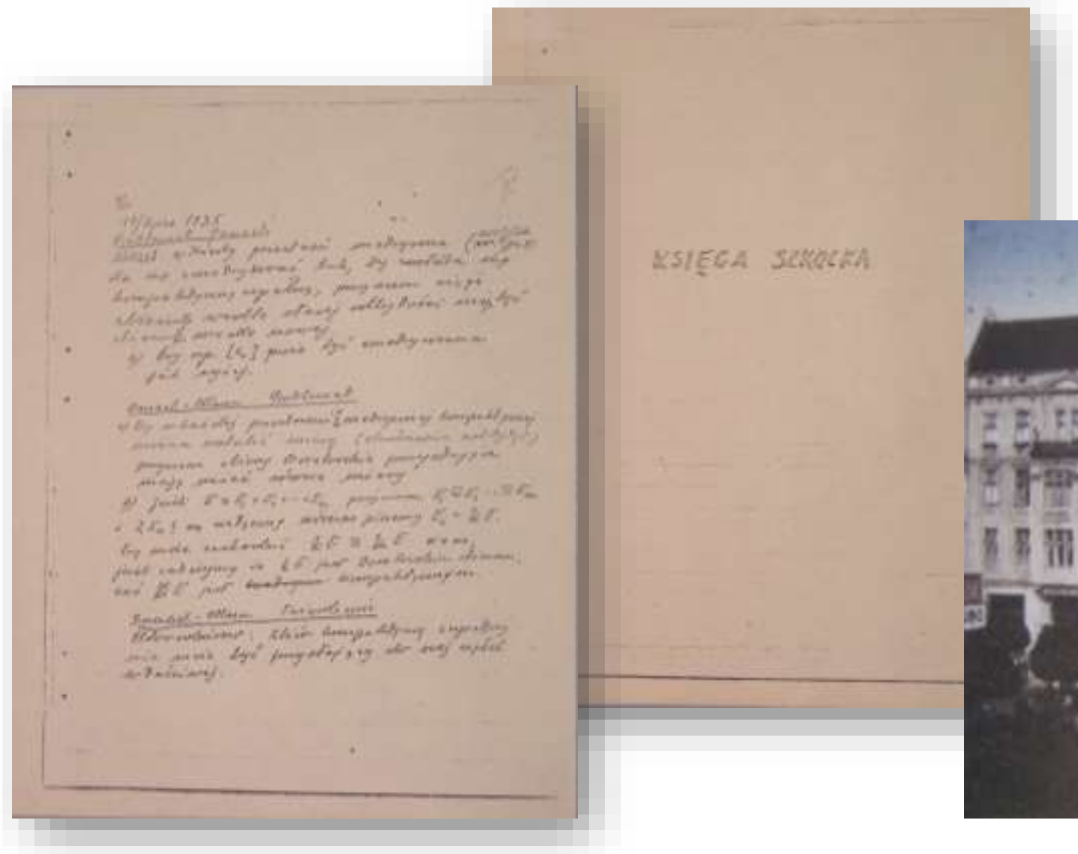
Juliusz Schauder
(1899 – 1943)

Stefan Banach
(1992 – 1945)

In 1930s the Faculty of Mathematics and Natural Sciences situated in the old building (U) of the university. After lectures and seminars mathematicians continued discussions in cafeteria near the university, in particular, in «Roma» (R) or «Scottish Café» (S).

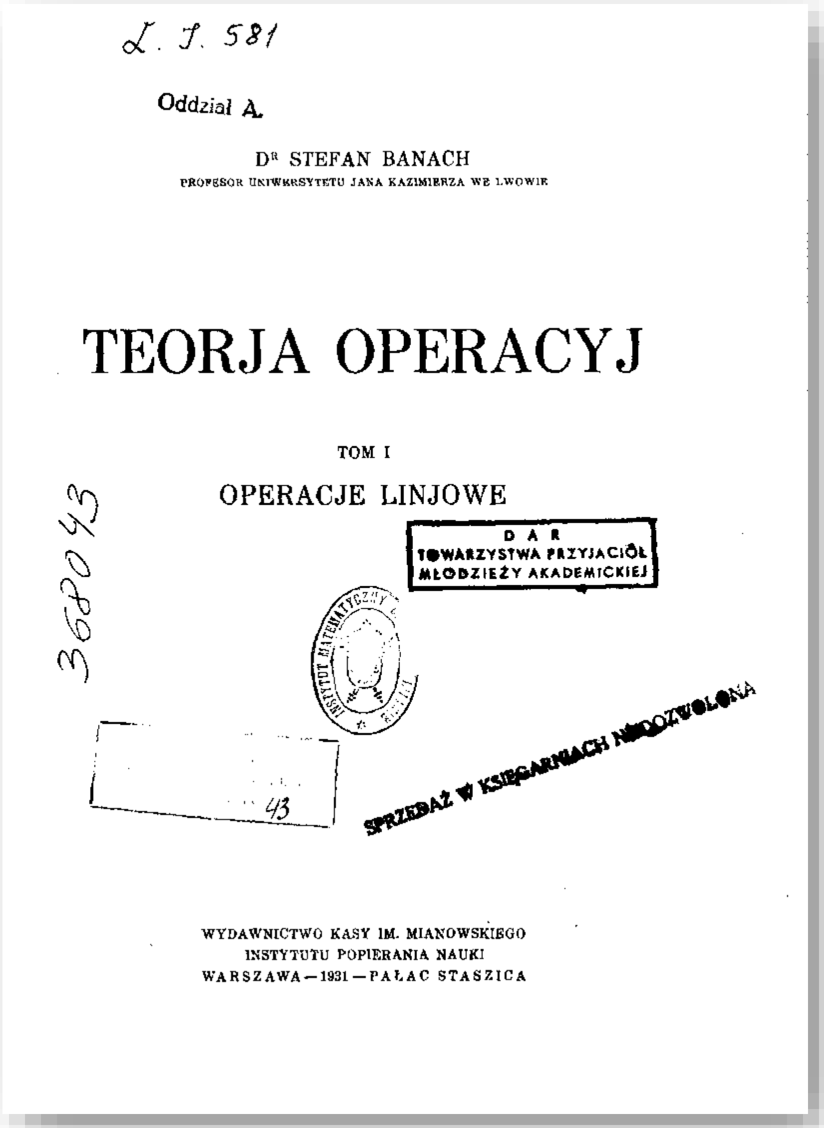


In 1935 they started to write open problems to a special notebook which became famous as «Księga Szkocka» («Scottish Book»).



U – University
R – «Roma»
S – Scottish Café
B – Banach spaces

The famous monograph of Stefan Banach was a cornerstone in creation of modern Functional Analysis:



L. I. 581

Oddział A.

Dr STEFAN BANACH

PROFESOR UNIWERSYTETU JANA KAZIMIERZA WE LWOWIE

TEORJA OPERACYJ

TOM I

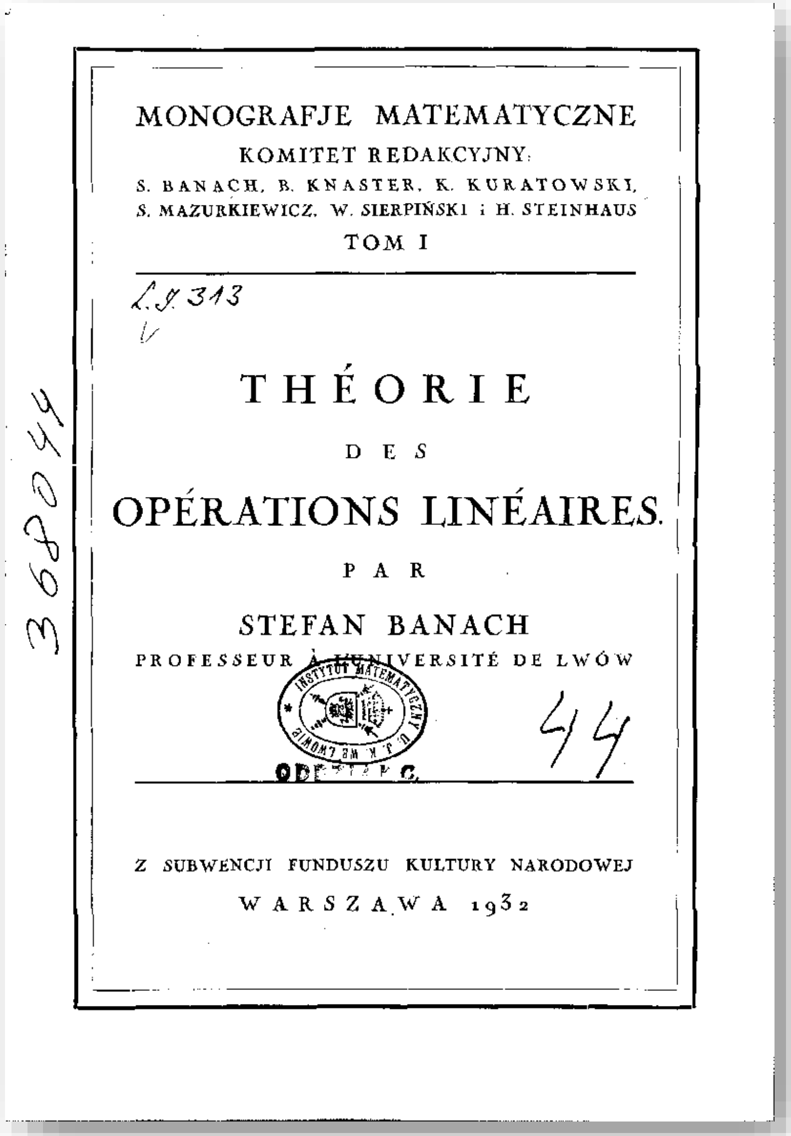
OPERACJE LINJOWE

368043

DAR TOWARZYSTWA PRZYJACIÓŁ MŁODZIEŻY AKADEMICKIEJ

SPRZEDAŻ W KSIĘGARNIACH NIEDOZWOLONA

WYDAWNICTWO KASY IM. MIANOWSKIEGO
INSTYTUTU POPIERANIA NAUKI
WARSZAWA — 1931 — PAŁAC STASZICA



MONOGRAFJE MATEMATYCZNE
KOMITET REDAKCYJNY:
S. BANACH, B. KNASTER, K. KURATOWSKI,
S. MAZURKIEWICZ, W. SIERPIŃSKI i H. STEINHAUS
TOM I

L. I. 313

368044

THÉORIE DES OPÉRATIONS LINÉAIRES

PAR

STEFAN BANACH

PROFESSEUR À L'UNIVERSITÉ DE LWÓW

44

Z SUBWENCJI FUNDUSZU KULTURY NARODOWEJ
WARSZAWA 1932

After the end of II WW Lviv became a part of USSR (more precisely, of Ukrainian Soviet Socialistic Republic). Polish mathematicians moved to various cities of Poland creating new mathematical centers (Wrocław, Warszawa, Łódź, Poznań) of Socialist RPL.

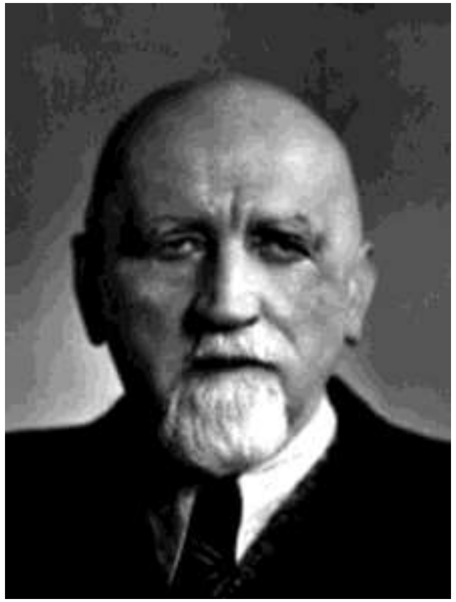
Nonetheless Banach's ideas continue to live and flourish in Lviv and Ukraine.

In 1948 Banach's monograph was translated (with the help of Myron Zarycki) into Ukrainian and became a standard textbook of Functional Analysis in the Soviet Union.

According to Professor Nikolskii (1905 **–** 2012), in the library of the mathematical faculty in MGU there was a queue of mathematicians that wanted to study Banach's book.

Myron Zarycki (1889 – 1961)

In spite of tectonic changes in Lviv mathematical landscape after the IIWW the seminar in Functional Analysis continued to work in the Lviv university. Among the active participants of the seminar we should mention:



Myron Zarycki
(1889 – 1961)

Yaroslav Lopatynskyi
(1906 – 1981)

Borys Gnedenko
(1912 – 1995)

Wladyslaw Lyantse
(1920 – 2007)

M.I. Kadets (1923 – 2011)

By a pure occasion, **Mikhail Kadets**, a young graduate of Kharkiv University, on his way to Makeevka (in Donetsk region) where he was assigned to work, bought Banach's «Course in Functional Analysis» and this determined his mathematical life and also the activity of his numerous successors. Since Kadets had no formal supervisor, he said that Banach's monograph was his «genuine supervisor».

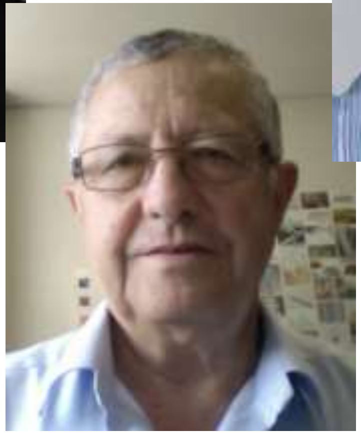
Banach's book contained many intriguing unsolved problems and Kadets got especially interested in one of them, namely the problem of topological equivalence of any two separable Banach spaces.

After many years of intensive efforts Kadets finally resolved it proving his elegant and now classical

Theorem (Kadets, 1967): *Any two separable infinite-dimensional Banach spaces are homeomorphic.*

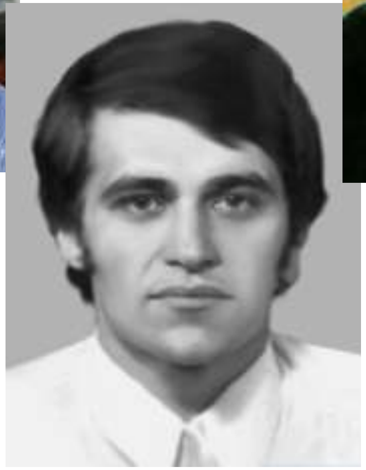
Mikhail Kadets had many pupils and followers which now are well-known mathematicians.

Vladimir Gurarii
1966



Stanimir Troyanski
1970

Stefan Heinrich
1976



Boris Godun
1978

Vladimir Fonf
1979

Mikahil Ostrovskii
1985

Vladimir Kadets
1985

Some of them participate at this conference.

Another powerful figure in development of Functional Analysis in Ukraine was **Mark Krein**, who created the schools of Functional Analysis in Odesa and Kyiv. The investigations of these schools were motivated by applied problems of Mechanics and Mathematical Physics. Among specialists in Geometry of Banach spaces Mark Krein is famous due to the Krein-Milman Theorem.
The genuine interests of Krein can be seen from the titles of his monographs:

Mark Krein (1907 – 1989)

Oscillation Matrices and Kernels and Small Vibrations of Mechanical Systems
(with F.R. Gantmacher), 1950. — 360 pp.

Introduction to the Theory of Linear Nonselfadjoint Operators in Hilbert Space
(with I.C. Gohberg), 1965. — 448 c.

Theory and Applications of Volterra Operators in Hilbert Space
(with I.C. Gohberg), 1967. — 508 c.

Stability of Solutions of Differential Equations in Banach Space
(with Yu.L. Daletskii), 1970. — 536 c.

The Markov Moment Problem and Extremal Problems
(with A.A. Nudel’man), 1973. — 551 c.

Introduction to the Spectral Theory of Operators in Spaces with Indefinite Metric
(with I.S. Iohvidov and H. Langer), 1982

According to Mathematics Genealogy Project, Mark Krein had **49** students and **949** descendants.

Mark Krein had a brother **Selim Krein,** a mathematician specializing in Applied Functional Analysis.
Together with Mark Krasnoselski (a student of Mark Krein) and Vladimir Sobolev,
Selim Krein created the famous School of Functional Analysis in Voronezh (Russia).

Selim Krein was a student of Mykola Boholyubov (who visited Lviv in 1940, wrote a problem to Scottish book, and was one of initiators of the translation of Banach's monographs into Ukrainian language).

One of **82** students (and **318** descendants) of Selim Krein was **Yuri Petunin**, a scientific teacher of **Anatolij Plichko**.

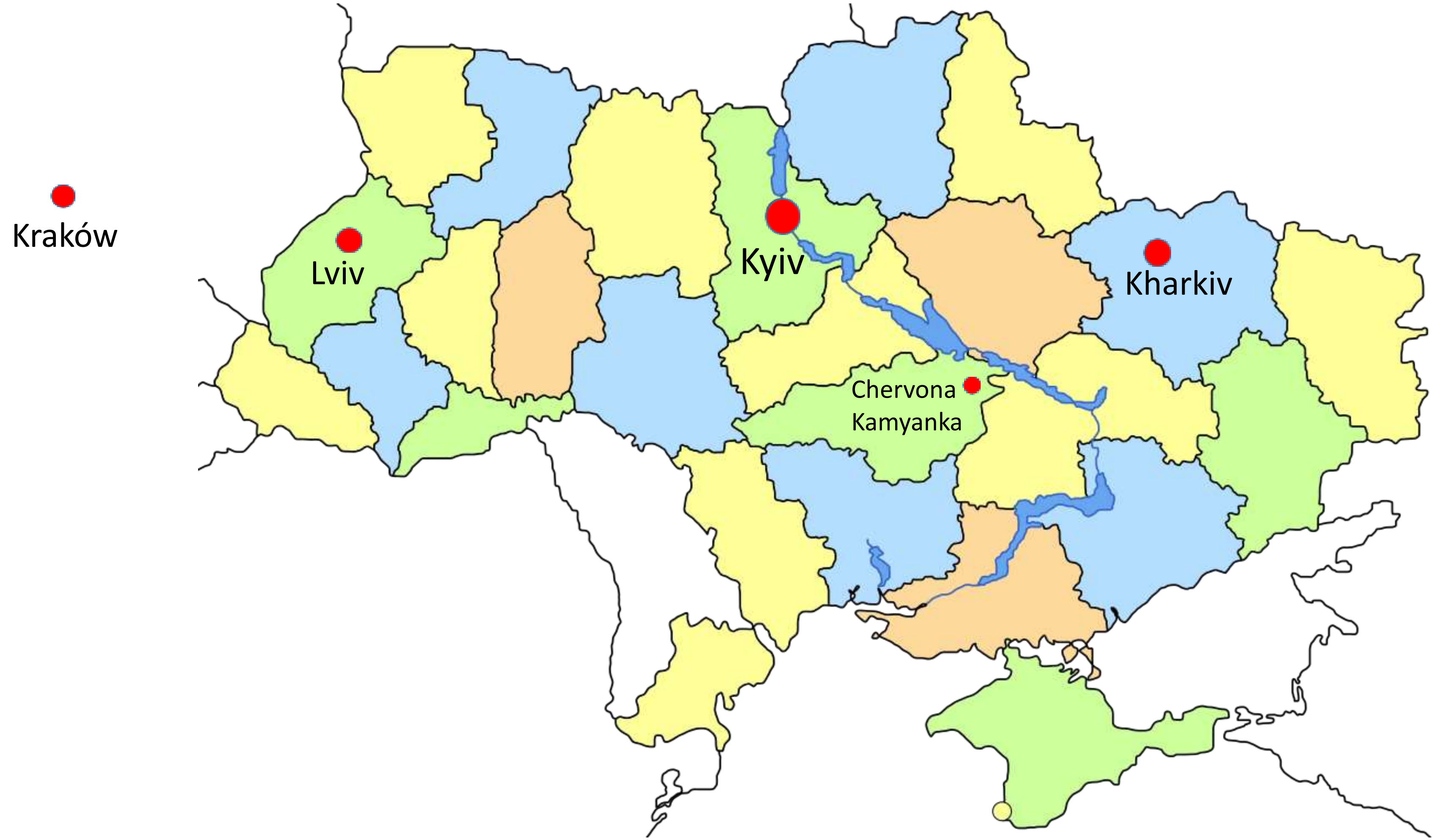
Mykola Bogolyubov
(1909 – 1992)

Selim Krein
(1917 – 1999)

Yuri Petunin
(1937 – 2011)

Anatoliy Plichko

Anatoly Plichko was born on 20 July 1949 in the village **Chervona Kamyanka** now of Kropyvnytsky region.
He studied in the mathematical school-internat in **Kyiv** and graduated from Kyiv University, where he made PhD.
Being interested in Banach spaces, Plichko kept contacts with M.I.Kadets and his group in **Kharkiv**.
During 1980 – 1995 Plichko worked in **Lviv** and afterwards in **Kraków**.



Anatolij Plichko has 5 defended PhD students:

- Volodymyr Maslyuchenko
- Mykhailo Popov
- Andriy Zagorodnyuk
- Andriy Razenkov
- Olga Kucher.

Three of them made habilitations and are active well-known mathematicians:

Volodymyr Maslyuchenko

Mykhailo Popov

Andriy Zagorodnyuk

The list of Plichko’s coauthors is much longer (over **50**) and includes many **participants** of this conference**:**

- G.A. Alexandrov
- **T.O. Banakh**
- T. Bartoszyński
- V.V. Buldygin
- **J.M.F. Castillo**
- E. Corbacho
- T. Dobrowolski
- M.N. Domanskii
- M. Dżamonja
- V.P. Fonf
- E.M. Galego
- **T. Gill**
- L.V. Gladun

- M. Gonzalez
- A.S. Granero
- L. Halbeisen
- M. Jimenes
- W.B. Johnson
- **V.M. Kadets**
- A. Kirtadze
- O.V. Kucher
- D. Kutzarova
- V.E. Lyantse
- L. Maligranda
- **E. Martin-Peinador**
- **V.K. Maslyuchenko**

- **I.K. Matsak**
- L.D. Menikhes
- **V. Montesinos**
- **J. Moreno**
- E. Murtinova
- **V. Mykhaylyuk**
- **M.I. Ostrovskii**
- G. Pantsulaia
- Yu.I. Petunin
- **M.M. Popov**
- **A.K. Prykarpatski**
- **Ya.G. Prytula**
- A. Razenkov

- N. Rusiashvili
- V.V. Shevchik
- I.Ja. Shneiberg
- **O.G. Storozh**
- **V. Tarieladze**
- P. Terenzi
- E.V. Tokarev
- F.S. Vakher
- V.A. Vinokurov
- **M. Wojtowicz**
- **D. Yost**
- **A.V. Zagorodnyuk**
- **M.M. Zarichnyi**

According to Pol Halmos,

Mathematicians sometimes classify themselves as either problem-solvers or theory-creators

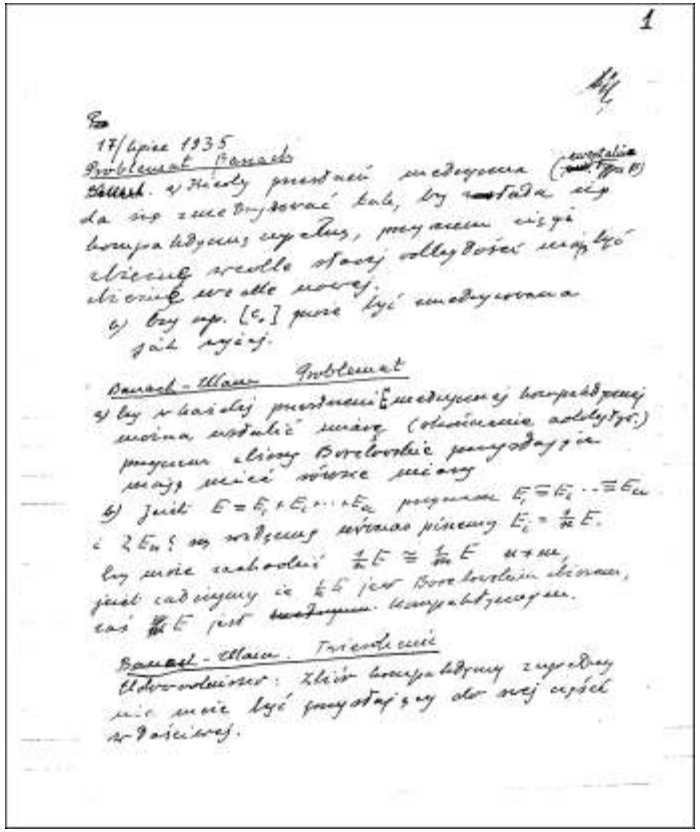
(I would add that there are also problem-creators :).

Anatolij Plichko definitely belongs to problem-solvers.

He (with coauthors) answered ten problems from the Scottish book.

Problem 1

(Banach)



УДК 515.12

ON A PROBLEM OF "SCOTTISH BOOK" CONCERNING CONDENSATIONS OF METRIC SPACES ONTO COMPACTA

T.O. Banakh, A.M. Plichko

T.O. Banakh, A.M. Plichko. *On a problem of "Scottish Book" concerning condensations of metric spaces onto compacta*, Matematychni Studii, **8**(1997) 119–122.

It is proved that every Banach space of density ω_1 admits a condensation onto the Hilbert cube.

1. Introduction

In this note we consider a question posed in 1935 in the known book of Lviv mathematicians [1]. In the modern terminology this question sounds as follows.

Problem (S. Banach). *When does a metric (possibly Banach) space X admit a condensation (i.e. a bijective continuous map) onto a compactum (= compact metric space)?*

We do not know the origins of this question. An obvious impulse for its appearance could be a well known result stating that every continuous image of

Problems
55 (Mazur),
56 (Mazur and Orlicz)
75 (Mazur):

On Automatic Continuity and Three Problems of "The Scottish Book" Concerning the Boundedness of Polynomial Functionals

Anatolij Plichko

Pedagogical University, Shevchenko 1, Kirovograd, 316050, Ukraine

and

Andriy Zagorodnyuk

Institute of Applied Problems of Mechanics and Mathematics, Naukova 3b, L'viv, 290053, Ukraine

Submitted by Richard M. Aron

Received July 7, 1997

In this paper we introduce and study the notions of isotropic mapping and essential kernel. In addition some theorems on the Borel graph and Baire mapping for polynomial operators are proved. It is shown that a polynomial functional from an infinite dimensional complex linear space into the field of complex numbers vanishes on some infinite dimensional affine subspace. © 1998 Academic Press

Problem 66
(Mazur)

On a problem of Mazur from "The Scottish Book" concerning second partial derivatives

Volodymyr Mykhaylyuk, Anatolij Plichko

Colloquium Mathematicum 141 (2015), 175-181
MSC: 26B05, 26B30.
DOI: 10.4064/cm141-2-3

Streszczenie

We comment on a problem of Mazur from "The Scottish Book" concerning second partial derivatives. We prove that if a function $f(x,y)$ of real variables defined on a rectangle has continuous derivative with respect to y and for almost all y the function $F_y(x) := f'_y(x,y)$ has finite variation, then almost everywhere on the rectangle the partial derivative f''_{yx} exists. We construct a separately twice differentiable function whose partial derivative f'_x is discontinuous with respect to the second variable on a set of positive measure. This solves the Mazur problem in the negative.

BANACH SPACES WITHOUT THE KADEC H-PROPERTY (SOLUTION OF A PROBLEM FROM THE "SCOTTISH BOOK")

A.M. Plichko

Dedicated to Professor V. Lyantse on his 75th birthday

Problem 89
(Mazur)

A. Plichko, *Banach spaces without the Kadec H-property (solution of a problem from the "Scottish Book")*, Matematychni Studii, **7**(1997) 59–60.

It is proved that if a separable Banach space X has not the Schur property then there exists an equivalent strictly convex norm on X without H-property.

In the book [2] S. Mazur rose the following question (Problem 89). "Let W be a convex body, located in the space (L^2), and such that its boundary W_b does not contain any interval; let $x_n \in W$, $(n = 1, 2, \ldots)$, $x_0 \in W_b$ and in addition let the sequence (x_n) converge weakly to x_0. Does then the sequence (x_n) converge strongly to x_0 ? It is known that this statement is true in the case where W is a sphere. Examine this problem for the case of other spaces."

Problems
51 (Mazur)
86 (Banach)
154 (Mazur)

ON THREE PROBLEMS FROM THE SCOTTISH BOOK CONNECTED WITH ORTHOGONAL SYSTEMS

BY

A. PLICHKO AND A. RAZENKOV (LVIV)

Introduction. In this paper we consider some questions connected with the following problems from [5]:

1. PROBLEM OF MAZUR ([5, Problem 154]): Let (φ_n) be an orthogonal system consisting of continuous functions and closed in C.

(a) If $f(t) \sim a_1\varphi_1(t) + a_2\varphi_2(t) + \ldots$ is the development of a given continuous function $f(t)$ and $n_1, n_2, \ldots$ denote the successive indices for which $a_{n_1} \neq 0, \ldots$, can one approximate $f(t)$ uniformly by linear combinations of the functions $\varphi_{n_1}(t), \varphi_{n_2}(t), \ldots$?

(b) Does there exist a linear summation method M such that the development of every continuous function $f(t)$ in the system $(\varphi_n(t))$ is uniformly summable by the method M to $f(t)$?

In [6] A. M. Olevskiĭ has given negative answers to both questions.

2. PROBLEM OF BANACH ([5, Problem 86]): Given a sequence of functions $(\varphi_n(t))$ which is orthogonal, normed, measurable, and uniformly bounded, can one always complete it, using functions with the same bound, to a sequence which is orthogonal, normed, and complete? Consider the case when infinitely many functions are necessary for completion.

This problem was first solved by S. Kaczmarz in [2]. Various solutions of this problem were found by B. S. Kashin, A. M. Olevskiĭ, S. V. Bochkarev and K. S. Kazarian [3, 4].

3. PROBLEM OF MAZUR ([5, Problem 51]):

a) Is every set of functions, measurable in $[0, 1]$ with the property that any two functions of the set are orthogonal, at most countable? (the functions are not assumed to be square-integrable!)

b) An analogous question for sequences: Is every set of sequences with the property that any two sequences (ε_n), (η_n) of this set are orthogonal, that is, $\sum_{n=1}^{\infty} \varepsilon_n\eta_n = 0$, at most countable?

Problem 188
(Eidelhait)

On a problem of Eidelheit from `The Scottish Book` concerning absolutely continuous functions

Lech Maligranda[a,*,1], Volodymyr V. Mykhaylyuk[b], Anatolij Plichko[c]

[a] Department of Mathematics, Luleå University of Technology, SE-971 87 Luleå, Sweden
[b] Department of Applied Mathematics, Chernivtsi National University, Kotsyubyn'skoho 2, 58012 Chernivtsi, Ukraine
[c] Institute of Mathematics, Cracow University of Technology, Warszawska 24, 31-155 Kraków, Poland

ARTICLE INFO

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Received 16 March 2010
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Submitted by Richard M. Aron

Dedicated to the memory of M. Eidelheit (1910–1943) on the occasion of 100th years of his birth

ABSTRACT

A negative solution of Problem 188 posed by Max Eidelheit in *the Scottish Book* concerning superpositions of separately absolutely continuous functions is presented. We discuss here this and some related problems which have also negative solutions. Finally, we give an explanation of such negative answers from the "embeddings of Banach spaces" point of view.

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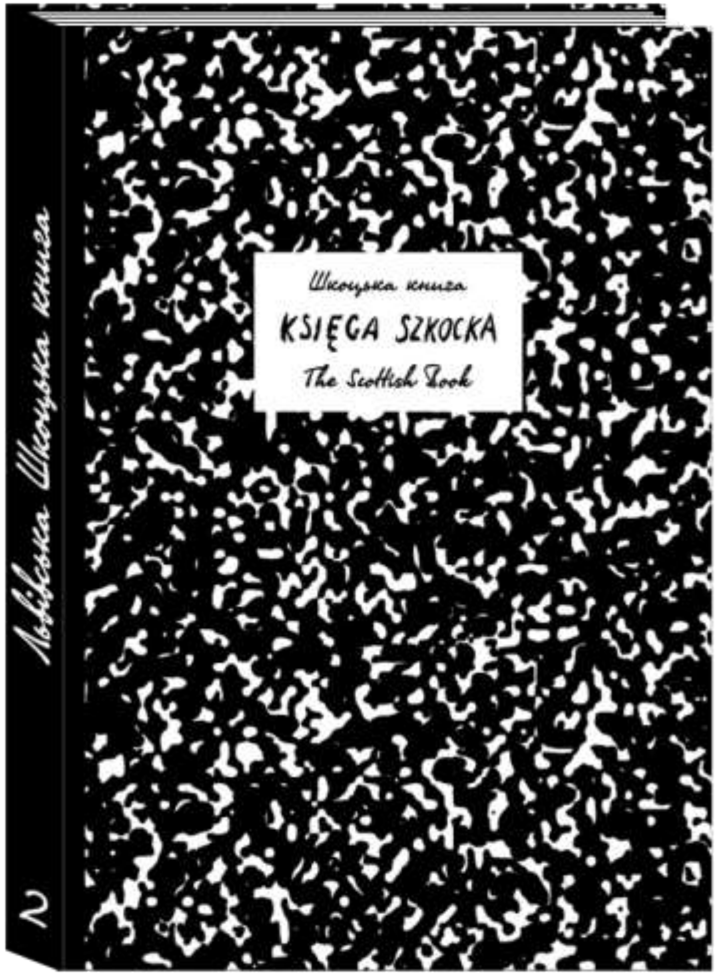
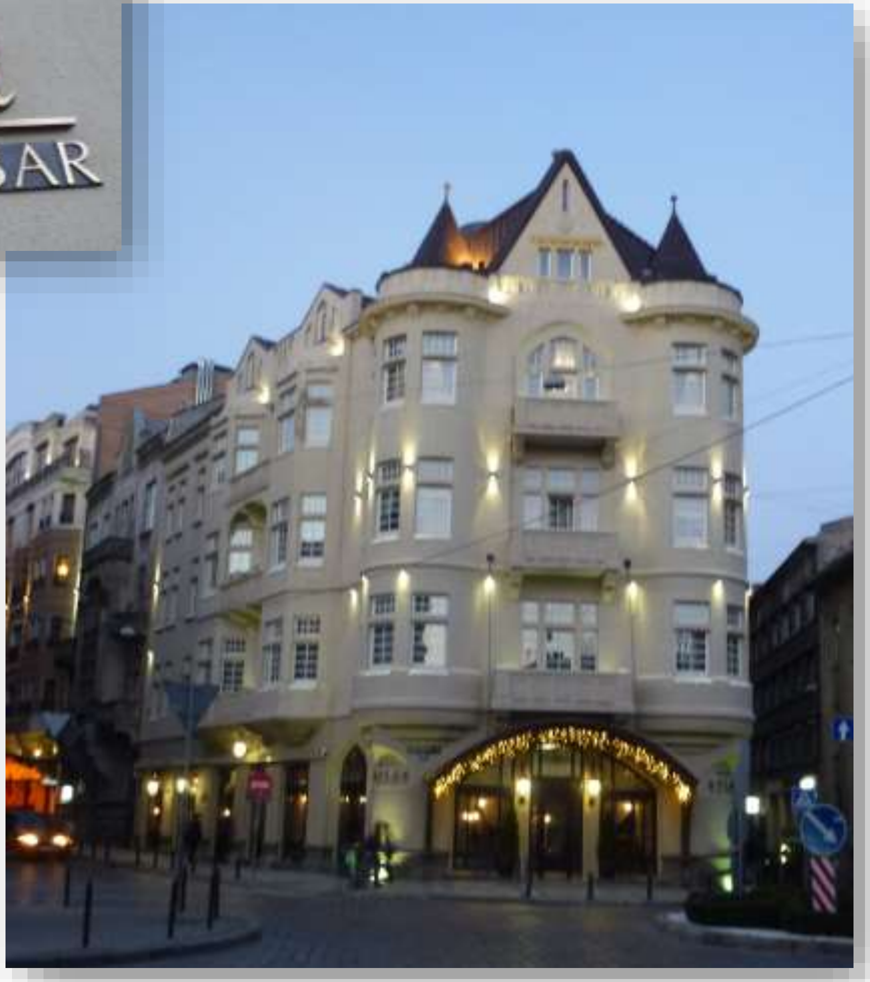
1. Introduction

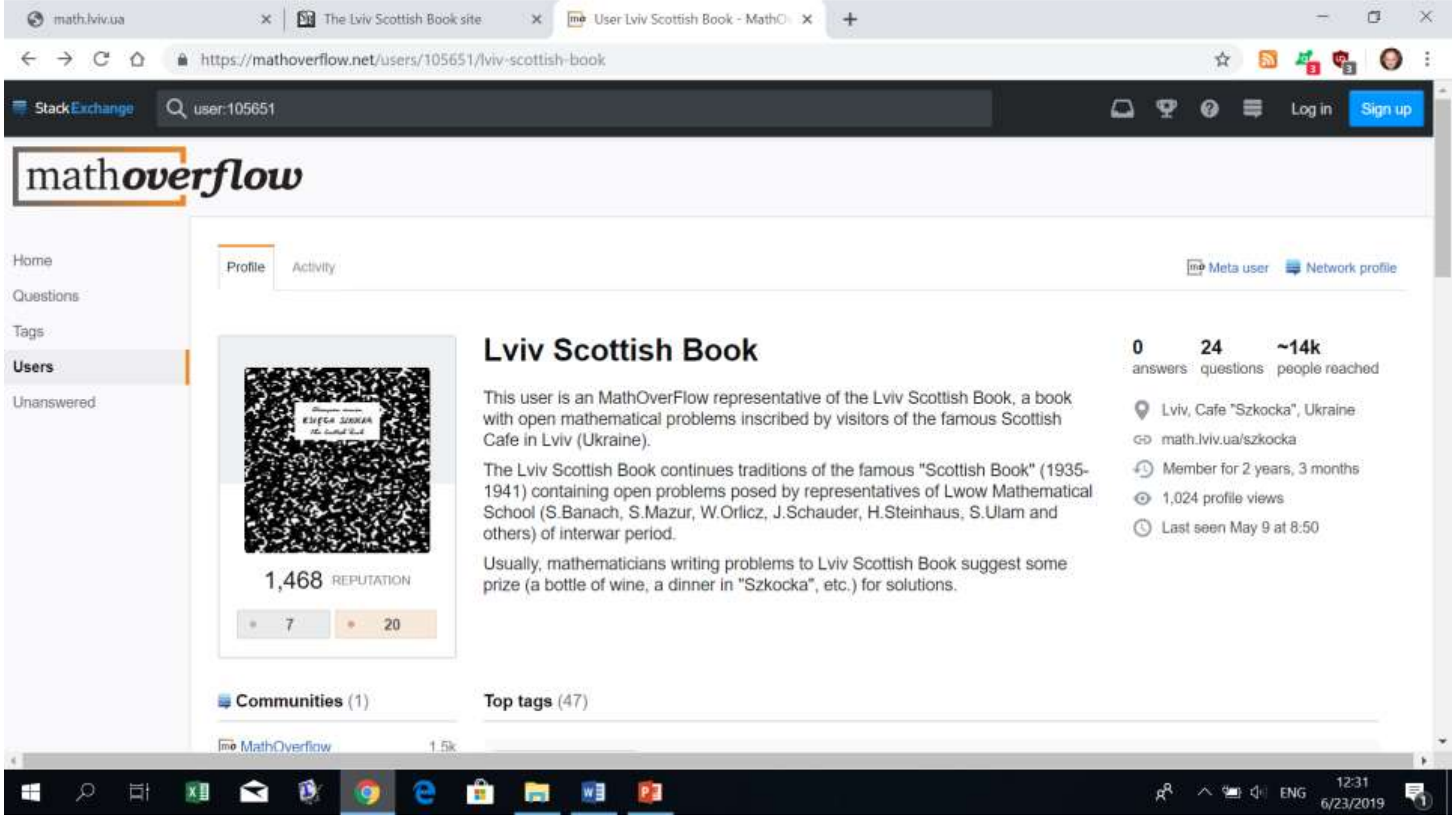
There are several equivalent definitions of the concept of absolute continuity. The notion and the term of *absolutely continuous* was introduced in 1905 by G. Vitali [27]. Let $I = [a, b] \subset \mathbb{R}$ and $f : I \to \mathbb{R}$. The function f is called *absolutely continuous on* I if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $a \leqslant a_1 < b_1 \leqslant a_2 < b_2 \leqslant \cdots \leqslant a_n < b_n \leqslant b$ the condition $\sum_{k=1}^{n}(b_k - a_k) < \delta$ implies that $\sum_{k=1}^{n}|f(b_k) - f(a_k)| < \varepsilon$ (cf. I.P. Natanson [14, p. 243]). Also we can say that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any finite collection of mutually disjoint intervals $I_k = (a_k, b_k) \subset I$ ($k = 1, 2, \ldots, n$) we have that $\sum_{k=1}^{n}|I_k| < \delta$ implies $\sum_{k=1}^{n}|f(b_k) - f(a_k)| < \varepsilon$. The requirement that the open intervals I_k are disjoint is sometimes stated by saying that the corresponding closed intervals $\{[a_k, b_k]\}_{k=1}^{n}$ must be *nonoverlapping*, that is, their interiors are disjoint. Note that since the number $n \in \mathbb{N}$ is arbitrary, we can also take $n = \infty$, that is, replace finite sums by series.

More information on solutions of problems from the Scottish Book by Ukrainian mathematicians can be found in the **Lviv Scottish Book,** available in the reopened **Café «Szkocka»** and also at **www.math.lviv.ua**

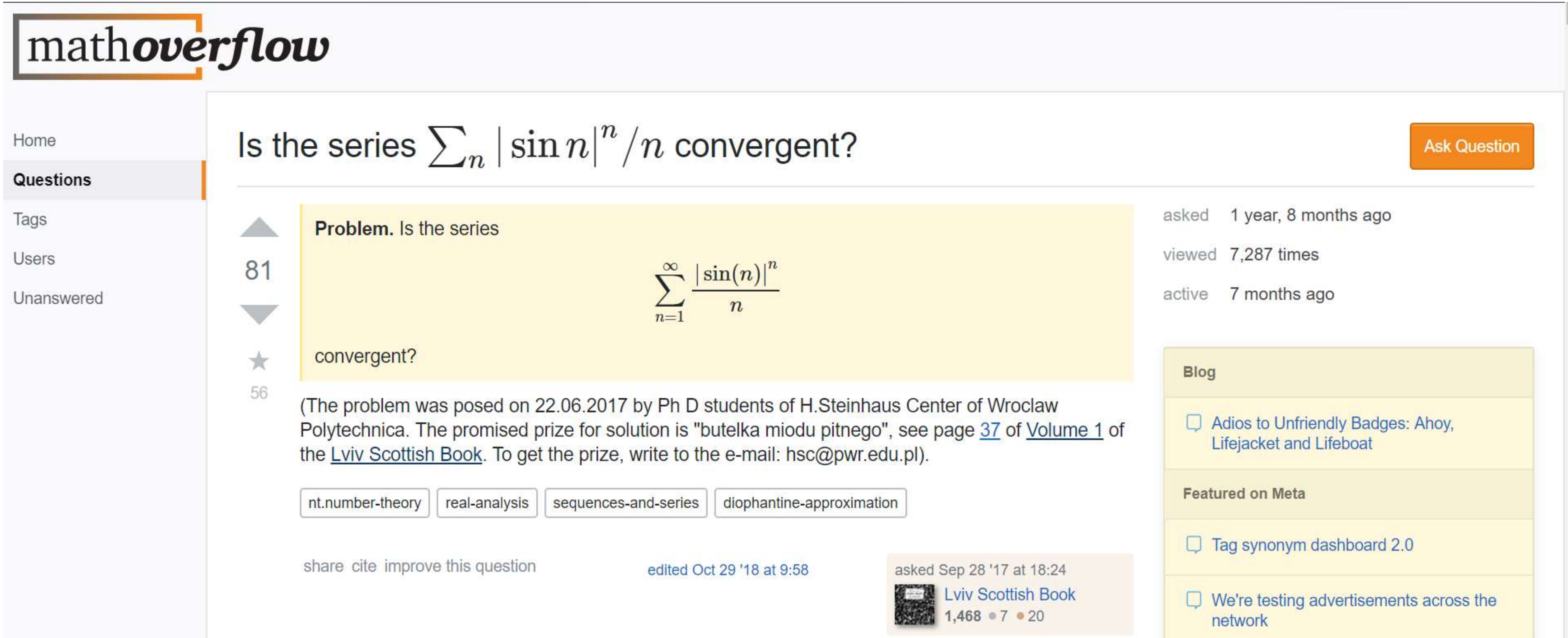




I encourage all the participants of the Conference to visit "Szkocka",
ask for the Lviv Scottish Book and write there your favorite open problems.
The problems are visible in the **Mathoverflow.net**,
where they have high chances to be discussed and eventually solved.



A bright recent example is the problem posed by Ph.D. students of Steinhaus Center form Wrocław:



mathoverflow

Home

Questions

Tags

Users

Unanswered

Is the series $\sum_n |\sin n|^n/n$ convergent?

Ask Question

81

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Problem. Is the series

$$\sum_{n=1}^{\infty} \frac{|\sin(n)|^n}{n}$$

convergent?

(The problem was posed on 22.06.2017 by Ph D students of H.Steinhaus Center of Wroclaw Polytechnica. The promised prize for solution is "butelka miodu pitnego", see page 37 of Volume 1 of the Lviv Scottish Book. To get the prize, write to the e-mail: hsc@pwr.edu.pl).

nt.number-theory | real-analysis | sequences-and-series | diophantine-approximation

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edited Oct 29 '18 at 9:58

asked Sep 28 '17 at 18:24

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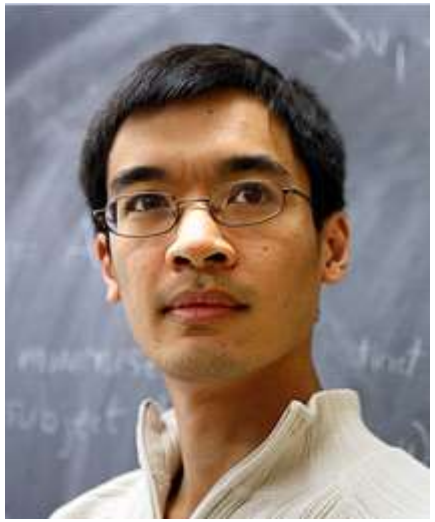
This problem was eventually solved by **Terence Tao**,
a mathematical protigy,
the Fields medalist of 2006.



LIFE AND TIMES OF TERENCE TAO

Age

7: Begins high school

9: Begins university

10,11,12: Competes in the International Mathematical Olympiads winning bronze, silver and gold medals

16: Honours degree from Flinders University

17: Masters degree from Flinders University

21: PhD from Princeton University

24: Professorship at University of California

31: Fields Medal

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Note that if π were rational (with even numerator), then $\sin(n)$ would equal 1 periodically, so the series would diverge. Similarly if π were a sufficiently strong Liouville number. Thus, to establish convergence, one must use some quantitative measure of the irrationality of π.

It is known that the irrationality measure μ of π is finite (indeed, the current best bound is $\mu \leq 7.60630853$). Thus, one has a lower bound

$$|\pi - \frac{p}{q}| \gg \frac{1}{q^{\mu+\varepsilon}}$$

for all p, q and any fixed $\varepsilon > 0$. This implies that

$$\mathrm{dist}(p/\pi, \mathbf{Z}) \gg \frac{1}{p^{\mu-1+\varepsilon}},$$

for all large p (apply the previous bound with q the nearest integer to p/π, multiply by q/π, and note that q is comparable to p). In particular, if $I \subset \mathbf{R}/\mathbf{Z}$ is an arc of length $0 < \delta < 1$, the set of n for which $n/\pi \bmod 1 \in I$ is $\gg \delta^{-1/(\mu-1+\varepsilon)}$-separated. This implies, for any natural number k, that the number of n in $[2^k, 2^{k+1}]$ such that $|\sin(n)|$ lies in any given interval J of length 2^{-k} (which forces $n/\pi \bmod 1$ to lie in the union of at most two intervals of length at most $O(2^{-k/2})$) is at most $\ll 2^{k(1-\frac{1}{2(\mu-1+\varepsilon)})}$, the key point being that this is a "power saving" over the trivial bound of 2^k. Noting (from Taylor expansion) that $|\sin(n)|^n \ll \exp(-j)$ if $n \in [2^k, 2^{k+1}]$ and $|\sin(n)| \in [1-\frac{j+1}{2^k}, 1-\frac{j}{2^k}]$, we conclude on summing in j that

$$\sum_{2^k \leq n < 2^{k+1}} |\sin(n)|^n \ll 2^{k(1-\frac{1}{2(\mu-1+\varepsilon)})}$$

and hence

$$\sum_{2^k \leq n < 2^{k+1}} \frac{|\sin(n)|^n}{n} \ll 2^{-k\frac{1}{2(\mu-1+\varepsilon)}}.$$

The geometric series on the RHS is summable in k, so the series $\sum_{n=1}^\infty \frac{|\sin(n)|^n}{n}$ is convergent. (In fact the argument also shows the stronger claim that $\sum_{n=1}^\infty \frac{|\sin(n)|^n}{n^{1-\frac{1}{2(\mu-1+\varepsilon)}}}$ is convergent for any $\varepsilon > 0$.)

EDIT: the apparent numerical divergence of the series may possibly be due to the reasonably good rational approximation $\pi \approx 22/7$, which is causing $|\sin(n)|$ to be close to 1 for n that are reasonably small odd multiples of 11. UPDATE: I now agree with Will that it is the growth of $-2^{3/2}/\pi^{1/2}n^{1/2}$, rather than any rational approximant to $1/\pi$, which was responsible for the apparent numerical divergence at medium values of n, as is made clear by the updated numerics on another answer to this question.

share cite improve this answer   edited Sep 30 '17 at 16:21   answered Sep 29 '17 at 2:06

Terry Tao
59k • 18 • 257 • 342

Thank you for your attention!

and